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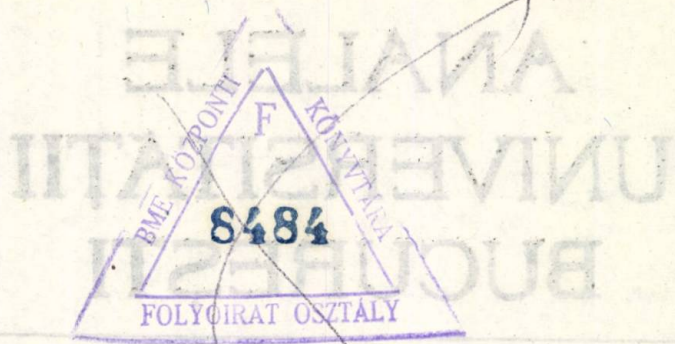
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SUMAR • SOMMAIRE •

CORNEA FLORIN OVIDIU, <i>Positive Definite Functions on Direct Sums of Locally Compact Abelian Groups</i>	3
GOGONEA SORIN, <i>Sur un problème de St. I. Gheorghitza</i>	7
LEIKO SVIATOSLAV, NICOLESCU LIVIU, <i>Des champs caractéristiques</i> . .	13
LICEA GABRIELA, NICULA VALENTIN, <i>On the Compensated inaccessible jumps of a Square Integrable Martingale</i>	23
MARCU DĂNUȚ, <i>No Tournament is Gradable</i>	27
MARCU DĂNUȚ, IORDACHE VIRGIL, <i>A Method for Finding the Offered Telephone Traffic from Flowed Telephone Traffic According to Erlang's Equation</i>	29
MATEESCU GEORGE-DANIEL, <i>On the Application of the Principle of Maximum Indetermination to the Finite Discrete Distribution</i>	35
NICOLESCU LIVIU, <i>Des champs spéciaux dans l'algèbre de déformation</i> . .	39
NGUYEN XUAN MY, <i>A note on N-Contextual Grammars</i>	49
ORFANIDIS ILIAS, <i>On Letter-Recognition in Greek Alphabet</i>	55
PREDA VASILE, <i>The Convergence of Weighting Process</i>	59
PURCARU ION, <i>An Entropic Measure of the Unilateral Dependence between Random Vectors</i>	65
TOMA ILEANA, <i>Solutions of Bilocal Polynomial Problems by Linearization</i> . .	71
TRANDAFIR ROMICĂ, <i>Détermination d'un intervalle contenant la solution de l'équation pour le principe de l'entropie maximale</i>	81
TUDORESCU CONSTANTIN, <i>Generative Systems with Growth Functions by Polynomial Type or Recursive Type</i>	87
ZHUANG XINGWU, <i>On Properties of Some Martingales</i>	111
VIAȚA FACULTĂȚII : 75 de ani de la nașterea Profesorului Gr. C. Moisil	119



SERIES MATHEMATICA

1981

STYAN • SOMMAIRE •

2	GONCALVES FLORES OVIDIO. Positive definite functions on direct sums of locally compact Abelian groups
7	GONZALEZ SORIN. Sur un probleme de J. G. Thompson
11	LENGO STATOSEAN, NICOLAESCU LIVIU. Les champs caracteristiques
23	ERL, GABRIELA, NICLA VALENTIN. On the Complemented indecomposable
37	MAHUT DANUT. Un tournoiement de Gerdorfs
49	MAHUT DANUT, IORDACHE VIRGIL. A method for finding the Offsets
59	MAHUT DANUT, IORDACHE VIRGIL. Telephone Traffic according to Erlang's
71	MAHUT DANUT, IORDACHE VIRGIL. On the foundation of the Principles of Mark
81	NICOLAESCU LIVIU. Des champs caracteristiques dans l'anneau de deformation
91	NICOLAESCU LIVIU. Des champs caracteristiques dans l'anneau de deformation
101	ONIANIDIS ILIAS. On Letter-Recognition in Greek Alphabet
111	PREDA VASILE. The Convergence of Working Process
121	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
131	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
141	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
151	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
161	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
171	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
181	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
191	PREDA VASILE. An Analytic Solution of the Uniform Dependence between
201	PREDA VASILE. An Analytic Solution of the Uniform Dependence between



POSITIVE DEFINITE FUNCTIONS ON DIRECT SUMS OF LOCALLY COMPACT ABELIAN GROUPS

FLORIN OVIDIU CORNEA

In this work we make some remarks concerning the generation of positive definite functions on direct sums of locally compact abelian groups.

In the sequel $(G, +)$ will be a locally compact abelian group and Γ will be the usual notation for the dual of G . The elements of Γ are actually the continuous morphisms from G to the unitary group $\{z \in \mathbb{C} \mid |z| = 1\}$. We recall that a positive definite function is a mapping $\varphi : G \rightarrow \mathbb{C}$ verifying the inequality

$$\sum_{i,j=1}^n C_i C_j \varphi(x_i - x_j) \geq 0$$

for all n -tuples $(C_1, \dots, C_n)$ of elements of $\mathbb{C}$ and for all n -tuples $(x_1, \dots, x_n)$ of elements of G .

We will use the wellknown theorem of Bochner (see for exemple [1]).

Theorem. Let $\varphi : G \rightarrow \mathbb{C}$ be a continuous function. The following properties are equivalent:

- i) φ is positive definite on G .
- ii) There exists a unique $\mu \in M_b^+(\Gamma)$ (a positive bounded Radon measure) such that

$$\varphi(x) = \int_{\Gamma} \rho(x) d\mu(\rho) \quad \text{for all } x \in G.$$

We will denote by $CP(G)$ the set of continuous positive definite functions on G . $CP(G)$ is obvious a convex cone.

Proposition 1. Let $(G_1, +), \dots, (G_n, +)$ be locally compact abelian groups and $f_1, \dots, f_n$ continuous functions on $G_1, \dots, G_n$ respectively such that $f_i(0) = 1$ for all $1 \leq i \leq n$. We define the function:

$$f_1 \otimes \dots \otimes f_n : G = \bigotimes_{i=1}^n G_i \rightarrow \mathbb{C}$$

$(f_1 \otimes \dots \otimes f_n)(x) = f_1(x_1) \dots f_n(x_n)$ where $x = x_1 + \dots + x_n$ is the unique writing of x as an element of G .

Then the following properties are equivalent :

- i) $f_1 \otimes \dots \otimes f_n$ is positive definite on G
- ii) f_n is positive definite on G for all $i \in \{1, \dots, n\}$.

Proof. i) $\Rightarrow$ ii) Let $1 \leq k \leq n$, $m \in \mathbf{N}$, $C_1, \dots, C_n \in \mathbf{C}$ and $x_1^{(k)}, \dots, x_m^{(k)} \in G_k$. Considering the vectors of G : $(0, \dots, 0, x_1^{(k)}, 0, \dots, 0), \dots, (0, \dots, 0, x_m^{(k)}, 0, \dots, 0)$ where $x_j^{(k)}$ is on the k position, and writing that $f_1 \otimes \dots \otimes f_n$ is positive definite on G we obtain :

$$\sum_{i,j=1}^n C_i \cdot \bar{C}_j (f_1 \otimes \dots \otimes f_n) ((0, \dots, 0, x_i^{(k)}, 0, \dots, 0) - (0, \dots, 0, x_j^{(k)}, 0, \dots, 0)) \geq 0.$$

Thus :

$$\sum_{i,j=1}^n C_i \cdot \bar{C}_j f_k \cdot (x_i^{(k)} - x_j^{(k)}) \geq 0$$

ii) $\Rightarrow$ i) Let Γ_i be the dual of G_i and Γ be the dual of G . Then $\Gamma = \Gamma_1 \oplus \dots \oplus \Gamma_n$ and for $\rho \in \Gamma$, $\rho(x) = \rho_1(x_1) \dots \rho_n(x_n)$ where $x = x_1 + \dots + x_n$.

Applying ii) and Bochner's theorem we obtain : there exists a unique $\mu_i \in M_b^+(\Gamma_i)$ such that

$$f_i(x) = \int_{\Gamma_i} \rho_i(x) d\mu_i(\rho_i) \quad \text{for all } x \in G_i \text{ and } i \in \{1, \dots, n\}.$$

Therefore :

$$\begin{aligned} (f_1 \otimes \dots \otimes f_n)(x) &= \left(\int_{\Gamma_1} \rho_1(x_1) d\mu_1(\rho_1) \right) \dots \left(\int_{\Gamma_n} \rho_n(x_n) d\mu_n(\rho_n) \right) = \\ &= \int_{\Gamma} \rho_1(x_1) \dots \rho_n(x_n) d(\mu_1 \otimes \dots \otimes \mu_n)(\rho_1, \dots, \rho_n) = \int_{\Gamma} \rho(x) d\mu(\rho) \end{aligned}$$

where $\mu = \mu_1 \otimes \dots \otimes \mu_n \in M_b^+(\Gamma)$. Invoking once more Bochner's theorem we obtain that $f_1 \otimes \dots \otimes f_n$ is positive definite on G .

Let $H \subset G$ be a closed subgroup. We denote $\Lambda = \{\rho \in \Gamma \mid \rho|_H = 1\}$. We remark that $\Lambda = \bigcap_{x \in H} \{\rho \in \Gamma \mid \rho(x) = 1\}$ and that ρ is continuous whence Λ is a closed subgroup of Γ . We denote $\widehat{G/H}$ the dual of G/H . The mapping $\Phi : \Lambda \rightarrow \widehat{G/H}$, $\Phi(\rho) = \bar{\rho}$, $\bar{\rho}(\bar{x}) = \rho(x)$ is well defined and is a group isomorphism. Therefore $\Lambda = \widehat{G/H}$.

Corollary 2. i) Let $f \in CP(G/H)$. There exists a unique $\mu \in M_b^+(\Lambda)$ such that $f(\bar{x}) = \int_{\Lambda} \rho(x) d\mu(\rho)$ for all $\bar{x} \in G/H$.

ii) Let $f \in CP(G/H)$ and $g \in CP(H)$. The mapping $f \otimes g : G \rightarrow \mathbf{C}$, $(f \otimes g)(\bar{x}, y) = f(\bar{x}) g(y)$ $\bar{x} \in G/H$, $y \in H$ is positive definite on G .

Proof. i) Let $f \in CP(G/H)$. By Bochner's theorem there exists a unique $\bar{\mu} \in M_b^+(\widehat{G/H})$ such that :

$$f(\bar{x}) = \int_{\widehat{G/H}} \bar{\rho}(\bar{x}) d\bar{\mu}(\bar{\rho})$$

We define $\mu(\rho) = \bar{\mu}(\bar{\rho})$. Obvious $\mu \in M_b^+(\Lambda)$ and is uniquely determined by $\bar{\mu}$. From $\Lambda = \widehat{G/H}$ we obtain :

$$f(\bar{x}) = \int_{\Lambda} \rho(x) d\mu(\rho)$$

ii) It is immediate from proposition 1.

Corollary 3 (a generalisation of Polya's theorem).

Let $f_1, \dots, f_n: \mathbb{R} \rightarrow [0, \infty)$ be continuous functions such that $f_i(0)=1$, f_i even, decreasing, convex for all $i \in \{1, \dots, n\}$. Then the function :

$$f_1 \otimes \dots \otimes f_n: \mathbb{R}^n \rightarrow [0, \infty)$$

$$(f_1 \otimes \dots \otimes f_n)(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n)$$

is positive definite on $\mathbb{R}^n$.

Proof. We apply the following theorem due to Polya : a continuous function $\varphi: \mathbb{R} \rightarrow [0, \infty)$ verifying : i) φ is even ; ii) φ is decreasing and convex on $(0, \infty)$, is positive definite and we take into account the proposition 1.

Proposition 4. Let $(G_1, +), \dots, (G_n, +)$ be locally compact abelian groups and $C = \bigoplus_{i=1}^n G_i$. Then for all $f \in CP(G)$ there exists a sequence of

the form $f_m = \sum_{i=1}^{k_m} \alpha_{i,m} g_{i,m}$ where $\alpha_{i,m} \in \mathbb{C}$, $g_{i,m}(x) = g_{i,m}(x_1 + \dots + x_n) = g_{i,m}^1(x_1) \dots g_{i,m}^n(x_n)$, $g_{i,m}^j$ being characters on G such that

$$f_m(x) \xrightarrow{m} f(x) \text{ for all } x \in G$$

Proof. Let $f \in CP(G)$. By Bochner's theorem there exists a unique $\mu \in M_b^+(\Gamma)$ such that

$$f(x) = \int_{\Gamma} \rho(x) d\mu(\rho)$$

We consider a sequence $\{\theta_m\}_m$, $\theta_m = \sum_{i=1}^{k_m} \alpha_{i,m} \cdot \varepsilon_{\{x^{i,m}\}}$ such that $\theta_m \rightarrow \mu$ vaguely, where for all $x \in G$, $\varepsilon_{\{x\}}$ is the Dirac measure concentrated in x .

Because we can choose the coefficients $\alpha_{i,m}$ such that $\sum_{i=1}^{k_m} \alpha_{i,m} \rightarrow \mu(1)$ it is clear that $\theta_m \rightarrow \mu$ in the Bernoulli topology (see [1]).

Since for all $\rho \in \Gamma$ and $x^i = (x_1^i, \dots, x_n^i) \in G$, $\varepsilon_{\{x^i\}}(\rho) = \rho(x^i) = \rho(x_1^i + \dots + x_n^i) = \rho_1(x_1^i) \dots \rho_n(x_n^i) = \varepsilon_{\{x_1^i\}}(\rho_1) \dots \varepsilon_{\{x_n^i\}}(\rho_n) = (\varepsilon_{\{x_1^i\}} \otimes \dots \otimes \varepsilon_{\{x_n^i\}})(\rho)$ and $\int_{\Gamma} \rho(x) d\varepsilon_{\{x^i\}}(\rho_k) = \rho_k(x^i)$ we obtain the desired result.

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SUR UN PROBLÈME DE ȘT. I. GHEORGHITZA

SORIN GOGONEA

1. Dans un travail publié en 1957, [1], Șt. I. Gheorghitza a donné la première forme du théorème généralisé du cercle qui, ultérieurement a joué un rôle central dans certains problèmes de la théorie de la filtration. Le problème posé en [1] est le suivant. Soit l'écoulement plan permanent dans un milieu poreux homogènes par portions, de la sorte que dans le domaine D_1 situé à l'extérieur de la circonférence $L: |z| = R$, du plan complexe $z = x + iy$, le coefficient de filtration est égal à k_1 , tandis que dans le domaine D_2 , intérieur à L , ce coefficient est égal à $k_2 \cdot k_1$ et k_2 sont des constantes distinctes données. Le mouvement est provoqué par un source située dans un point $z_0 \in D_1$ quelconque et ayant le débit égal à q .

Si on admet la validité de la loi de filtration de Darcy alors le problème revient à la détermination des potentiels complexes $f_j(z) = \varphi_j(x, y) + i\psi_j(x, y)$, $j = 1, 2$. $f_1(z)$ est définie en D_1 sauf z_0 où elle possède une singularité logarithmique, tandis que $f_2(z)$ est définie et régulière en D_2 . Sur L sont valables les conditions de raccordement

$$(1) \quad \operatorname{Re} \{k_2 f_1(\zeta) - k_1 f_2(\zeta)\} = 0; \quad \operatorname{Im} \{f_1(\zeta) - f_2(\zeta)\} = 0; \quad \zeta \in L$$

Șt. I. Gheorghitza a résolu le problème par une méthode directe en utilisant des développements en séries des potentiels. Par sommation des séries respectives la solution a résulté sous la forme.

$$(2) \quad \begin{aligned} f_1(z) &= \frac{9}{2\pi} \left[\ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln \left(\frac{R^2}{z} - \bar{z}_0 \right) \right] \\ f_2(z) &= \frac{9}{2\pi} \frac{2k_2}{k_1 + k_2} \ln(z - z_0) \end{aligned}$$

Si on denote $f_0(z) = \frac{9}{2\pi} \ln(z - z_0)$ alors on obtient de (2)

$$(3) \quad \begin{aligned} f_1(z) &= f_0(z) + \frac{k_1 - k_2}{k_1 + k_2} f_0 \left(\frac{R^2}{z} \right) \\ f_2(z) &= \frac{2k_2}{k_1 + k_2} f_0(z) \end{aligned}$$

On constate d'ailleurs aisément que (3) donne l'expression des potentiels complexes pour n'importe quelle fonction $f_0(z)$, à cela près que toutes les singularités de cette fonction se trouvent seulement en D_1 .

Dans [2], C. Jacob a obtenu la solution du problème en question dans un cas plus général, en admettant qu'on peut avoir des singularités tant en D_1 que dans D_2 . La solution respective n'est pas valable dans le cas des singularités logarithmiques situées à l'intérieur de la circonférence L . La solution dans ce cas a été donnée en [3] (voir également [4]). Nous avons obtenu plus tard la solution dans le cas où l'on admet des singularités sur la frontière L , [5].

2. Dans ce travail nous allons considérer un problème qui généralise le problème initial de St. I. Gheorghitza. À savoir on étudie le mouvement plan provoqué par une source dans un milieu poreux homogène par portion ayant la forme comme dans la fig. 1. Les domaines Ω_j où le coefficient de filtrations a les valeurs constantes k_j , $j = 1, 2, 3, 4$, sont définis par

$$\Omega_1: |z| > R, y > 0; \quad \Omega_2: |z| > R, y < 0$$

$$\Omega_3: |z| < R, y > 0; \quad \Omega_4: |z| < R, y < 0$$

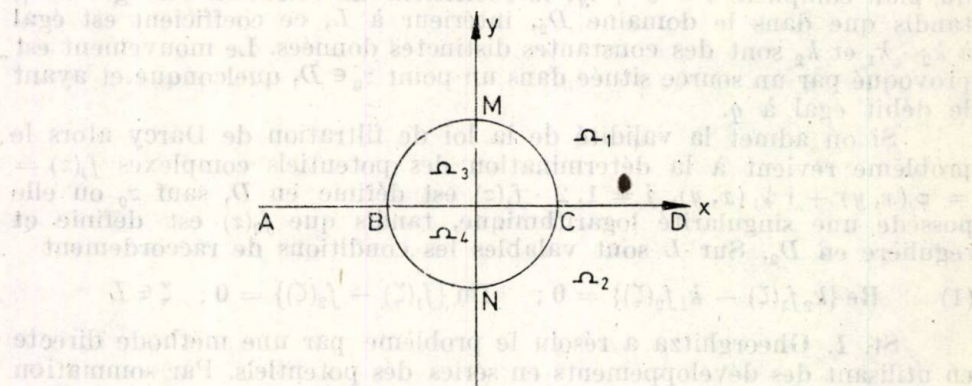


Fig. 1

Soient $f_j(z)$ le potentiel complexe en Ω_j et $F_j(z)$ la fonction qui définit les singularités de $f_j(z)$. Alors la différence $f_j(z) - F_j(z)$ est régulière en Ω et sur les frontières on doit satisfaire aux conditions de raccordement

$$\begin{aligned} \text{Re } \{k_2 f_1(\zeta) - k_1 f_2(\zeta)\} &= 0; \quad \text{Im } \{f_1(\zeta) - f_2(\zeta)\} = 0, \text{ sur } AB \cup CD \\ \text{Re } \{k_3 f_4(\zeta) - k_4 f_3(\zeta)\} &= 0; \quad \text{Im } \{f_3(\zeta) - f_4(\zeta)\} = 0, \text{ sur } BC \\ (4) \quad \text{Re } \{k_3 f_1(\zeta) - k_1 f_3(\zeta)\} &= 0; \quad \text{Im } \{f_1(\zeta) - f_3(\zeta)\} = 0, \text{ sur } \widehat{BMC} \\ \text{Re } \{k_2 f_4(\zeta) - k_4 f_2(\zeta)\} &= 0; \quad \text{Im } \{f_2(\zeta) - f_4(\zeta)\} = 0, \text{ sur } \widehat{BNC} \end{aligned}$$

Ce problème a été traité sous une forme générale en [6]. La méthode de résolution développée, qui utilise certaines idées de nos travaux antérieurs, et qui permet d'obtenir la solution sous une forme simple, exige qu'on prend

$$(5) \quad k_1 k_4 = k_2 k_3$$

En conséquence seulement trois coefficients de filtration sont indépendants. Le relation (5) est nécessaire et suffisante pour la résolution du problème en suivant la méthode imaginée en [6]. Mais il n'en résulte que ce soit une condition nécessaire pour obtenir la solution par un autre procédé. Le problème de la signification de la condition (5) reste encore ouvert.

Les particularités de l'application de la méthode de [6] dans le cas des singularités logarithmiques résultent du fait que la condition que $F_j(z)$ ait des singularités seulement en Ω_j n'est jamais respectée. En effet

si $z_0 \in \Omega_1$ (ou bien $z_0 \in \Omega_2$) alors $F_1(z) = \frac{9}{2\pi} \ln(z - z_0)$ (ou $F_2(z) = \frac{9}{2\pi} \ln(z - z_0)$) possède une singularité à l'infini qui est un point

frontière de Ω_1 et Ω_2 . Si $z_0 \in \Omega_3$, $F_3(z) = \frac{9}{2\pi} \ln(z - z_0)$ possède également une singularité à l'infini. La même conclusion vaut si $z_0 \in \Omega_4$.

Afin de dépasser cette difficulté remarquons, comme dans [3], que la solution générale de [6] vérifie les conditions de raccordement même si les singularités de $F_j(z)$ sont situées arbitrairement. Alors, nous allons introduire certaines singularités logarithmiques supplémentaires de la sorte que d'une côté les singularités qui ne correspondent pas au problème donnée disparaissent et de l'autre côté on assure la condition relativement au débit.

3. Supposons que $z_0 \in \Omega_1$, les raisonnements pour $z_0 \in \Omega_2$ étant les mêmes. On prend donc $F_1(z) = \frac{9}{2\pi} \ln(z - z_0)$, $F_3(z) = F_4(z) = 0$.

Comme nous avons déjà observé on ne peut prendre d'avance $F_2(z) = 0$. Il est d'ailleurs clair que si le fluide sort, par exemple, de z_0 il s'écoule vers l'infini et donc on doit avoir une singularité logarithmique pour $z = \infty$. Nous prendrons donc $F_2(z) = \frac{\lambda 9}{2\pi} \ln \frac{z}{R}$ où λ est une constante réelle qui doit être déterminée. Alors les formules de [6] donnent pour $f_1(z)$ et $f_2(z)$

$$(6) \quad \begin{aligned} f_1(z) = & \frac{9}{2\pi} \left\{ \ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln(z - \bar{z}_0) + \frac{2k_1}{k_1 + k_2} \lambda \ln \frac{z}{R} + \right. \\ & + \frac{k_1 - k_3}{k_1 + k_3} \ln \left(\frac{R^2}{z} - \bar{z}_0 \right) + \frac{(k_1 - k_3)(k_1 - k_2)}{(k_1 + k_3)(k_1 + k_2)} \ln \left(\frac{R^2}{z} - z_0 \right) + \\ & \left. + \frac{2k_1(k_1 - k_3)}{(k_1 + k_2)(k_1 + k_3)} \lambda \ln \frac{R}{z} \right\} \end{aligned}$$

$$f_2(z) = \frac{9}{2\pi} \left\{ \lambda \ln \frac{z}{R} + \frac{k_2 - k_1}{k_1 + k_2} \lambda \ln \frac{z}{R} + \frac{2k_2}{k_1 + k_2} \ln(z - z_0) + \right. \\ \left. + \frac{k_2 - k_4}{k_2 - k_4} \lambda \ln \frac{z}{R} + \frac{(k_2 - k_4)(k_2 - k_1)}{(k_1 + k_2)(k_2 + k_4)} \lambda \ln \frac{R}{z} + \right. \\ \left. + \frac{2k_2(k_2 - k_4)}{(k_1 + k_2)(k_2 + k_4)} \ln \left(\frac{R^2}{z} - \bar{z}_0 \right) \right\}$$

Le débit à travers d'une courbe fermée quelconque Γ située au voisinage de $z = \infty$ doit être égal à q . Si $\Gamma_1 = \Omega_1 \cap \Gamma$ et $\Gamma_2 = \Omega_2 \cap \Gamma$ on a, à la suite des formules bien connues

$$(17) \quad \operatorname{Im} \left\{ \int_{\Gamma_1} f_1(\zeta) d\zeta \right\} + \operatorname{Im} \left\{ \int_{\Gamma_2} f_2(\zeta) d\zeta \right\} = q$$

Compte tenu qu'au voisinage de $z = \infty$ on a

$$(8) \quad \begin{aligned} f_1(z) &= \alpha \ln z + \alpha_0 + \frac{\alpha_1}{z} + \dots \\ f_2(z) &= \beta \ln z + \beta_0 + \frac{\beta_1}{z} + \dots \end{aligned}$$

et on suppose que Γ est une circonférence dont le rayon tend vers l'infini on obtient $(\alpha + \beta)\pi = q$. Vu les expressions de α et β on déduit aisément que $\lambda = 0$. Alors compte tenu des expressions de $f_3(z)$ et $f_4(z)$ données en [6] on obtient la solution du problème posé sous la forme

$$(9) \quad \begin{aligned} f_1(z) &= \frac{9}{2\pi} \left\{ \ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln(z - z_0) + \frac{k_1 - k_3}{k_1 + k_3} \ln \left(\frac{R^2}{z} - \bar{z}_0 \right) + \right. \\ &\quad \left. + \frac{(k_1 - k_3)(k_1 - k_2)}{(k_1 + k_3)(k_1 + k_2)} \ln \left(\frac{R^2}{z} - z_0 \right) \right\} \\ f_2(z) &= \frac{9}{\pi} \frac{k_2}{k_1 + k_2} \left\{ \ln(z - z_0) + \frac{k_2 - k_4}{k_2 + k_4} \ln \left(\frac{R^2}{z} - \bar{z}_0 \right) \right\} \\ f_3(z) &= \frac{9}{\pi} \frac{k_3}{k_1 + k_3} \left\{ \ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln(z - \bar{z}_0) \right\} \\ f_4(z) &= \frac{9}{\pi} \frac{2k_2 k_4}{(k_1 + k_2)(k_2 + k_4)} \ln(z - z_0) \end{aligned}$$

4. En particulier si $k_1 = k_2$ il résulte de (6) que $k_3 = k_4$. On voit aisément que $f_1(z) = f_2(z) = f_1^*(z)$ et $f_3(z) = f_4(z) = f_3^*(z)$, où $f_1^*(z)$ et $f_3^*(z)$ coïncident avec la solution (2) de St. I. Gheorghitza, pour $D_1 = |z| > R$ et $D_3 : |z| < R$.

Un autre cas particulier remarquable s'obtient pour $k_3 = k_4 = 0$. La condition (5) est satisfaite et on obtient la solution correspondante au cas de l'écoulement autour du cylindre impérmeable $|z| = R$ situé dans un milieu poreux ayant le coefficient de filtration égal respectivement à k_1 et k_2 en Ω_1 et Ω_2 .

$$(10) \quad \begin{aligned} f_1(z) = \frac{9}{2\pi} & \left\{ \ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln(z - \bar{z}_0) + \ln\left(\frac{R^2}{z} - \bar{z}_0\right) + \right. \\ & \left. + \frac{k_1 - k_2}{k_1 + k_2} \ln\left(\frac{R^2}{z} - z_0\right) \right\} \end{aligned}$$

$$f_2(z) = \frac{9}{\pi} \frac{k_2}{k_1 + k_2} \left\{ \ln(z - z_0) + \ln\left(\frac{R^2}{z} - \bar{z}_0\right) \right\}$$

$$f_3(z) = f_4(z) = 0$$

5. Soit maintenant que la source se trouve en $z_0 \in \Omega_3$, c'est-à-dire $F_3(z) = \frac{9}{2\pi} \ln(z - z_0)$. Cette fonction a une singularité à l'infini. D'une autre côté dans la solution générale de [6] il existe un terme de la forme $F_3\left(\frac{R^2}{\bar{z}}\right)$ qui introduit une singularité à l'origine. Nous sommes conduit de la sorte de prendre $F_1(z) = \frac{9\tilde{\lambda}}{2\pi} \ln \frac{z}{R}$, $\tilde{\lambda}$ étant une constante et ensuite $F_2(z) = F_4(z) = 0$. Ecrivons pour l'instant les expressions de $f_3(z)$ et $f_4(z)$ qui résultent de [6].

$$\begin{aligned} f_3(z) = \frac{9}{2\pi} & \left\{ \ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln(z - \bar{z}_0) + \frac{2k_3}{k_1 + k_3} \tilde{\lambda} \ln \frac{z}{R} + \right. \\ & + \frac{k_3 - k_1}{k_1 + k_3} \ln\left(\frac{R^2}{z} - \bar{z}_0\right) + \frac{(k_3 - k_1)(k_1 - k_2)}{(k_1 + k_2)(k_1 + k_3)} \ln\left(\frac{R^2}{z} - z_0\right) + \\ & \left. + \frac{2k_3(k_1 - k_2)}{(k_1 + k_2)(k_1 + k_3)} \tilde{\lambda} \ln \frac{z}{R} \right\} \end{aligned}$$

$$\begin{aligned} f_4(z) = \frac{9}{2\pi} & \left\{ \frac{2k_2}{k_1 + k_2} \ln(z - z_0) + \frac{4k_2 k_4}{(k_1 + k_2)(k_2 + k_4)} \tilde{\lambda} \ln \frac{z}{R} + \right. \\ & \left. + \frac{2k_2(k_4 - k_2)}{(k_1 + k_2)(k_2 + k_4)} \ln\left(\frac{R^2}{z} - \bar{z}_0\right) \right\} \end{aligned}$$

Pour que la singularité à l'origine disparaisse il faut et il suffit que le coefficient de $\ln z$ soit nul. On constate aisément qu'on doit prendre dans tous les deux cas $\tilde{\lambda} = \frac{k_3 - k_1}{2k_3}$. Alors en utilisant aussi les expressions de $f_1(z)$ et $f_2(z)$ de [6], on obtien, à la suite des calculs simples, la solution du problème posé sous la forme

$$f_1(z) = \frac{9}{\pi} \frac{k_1}{k_1 + k_3} \left\{ \ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln(z - \bar{z}_0) + \frac{k_3 - k_1}{k_1 + k_2} \ln \frac{z}{R} \right\}$$

$$f_2(z) = \frac{9}{\pi} \frac{k_2}{(k_1 + k_2)(k_2 + k_4)} \left\{ 2k_2 \ln(z - z_0) + \frac{k_4}{k_3} (k_3 - k_1) \ln \frac{z}{R} \right\}$$

$$f_3(z) = \frac{9}{2\pi} \left\{ \ln(z - z_0) + \frac{k_1 - k_2}{k_1 + k_2} \ln(z - \bar{z}_0) + \right. \\ \left. + \frac{k_3 - k_1}{k_1 + k_3} \ln \frac{R^2 - z\bar{z}_0}{R} + \frac{(k_3 - k_1)(k_1 - k_2)}{(k_1 + k_2)(k_1 + k_3)} \ln \frac{R^2 - z\bar{z}_0}{R} \right\}$$

$$f_4(z) = \frac{9}{\pi} \frac{k_2}{k_1 + k_2} \left\{ \ln(z - z_0) + \frac{k_4 - k_2}{k_2 + k_4} \ln \frac{R^2 - z\bar{z}_0}{R} \right\}.$$

On peut vérifier sans difficultés, en utilisant une relation de type (7), que le débit qui passe par une courbe fermée située au voisinage de $z = \infty$ est précisément égal à q .

En particulier si $k_1 = k_2$ et $k_3 = k_4$, égalités compatibles avec (6), on retrouve la solution que nous avons donné auparavant, [3], [4].

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DES CHAMPS ($\overset{1}{F}, \dots, \overset{m}{F}$)-CARACTÉRISTIQUES

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§0 Introduction

Les variétés différentiables, les applications différentiables, les champs tensoriels et les connexions linéaires qui interviennent à la suite sont supposés de la classe C^∞ .

Soit M une variété différentiable à n dimensions. On note par $\mathcal{F}(M)$ — l'anneau des fonctions réelles, différentiables, définies sur M et par $\mathcal{T}_s^r(M)$ — le $\mathcal{F}(M)$ -module des champs de tenseurs du type (r, s) sur M . Par $\delta \in \mathcal{T}_1^1(M)$ nous notons le tenseur de Kronecker.

§1. Des courbes $(\nabla, \overset{1}{F}, \dots, \overset{m}{F})$ -planes.

Considérons une connexion linéaire ∇ sur M et soit $\overset{1}{F}, \dots, \overset{m}{F} \in \mathcal{T}_1^1(M)$. Soit $C: (a, b) \rightarrow M$ une courbe régulière et soit $\dot{C} = C_* \left(\frac{d}{dt} \right)$, le champ vectoriel tangent à la courbe C (avec C_* on a noté la différentielle de l'application C).

Définition 1.1. La courbe $C: (a, b) \rightarrow M$ s'appelle $(\nabla, \overset{1}{F}, \dots, \overset{m}{F})$ -plane s'il existe $m+1$ fonction $h, \overset{1}{h}, \dots, \overset{m}{h}: (a, b) \rightarrow R$ ainsi que :

$$(1.1) \quad \nabla_{\dot{C}} \dot{C} = h \dot{C} + \overset{11}{h} F(\dot{C}) + \dots + \overset{mm}{h} F(\dot{C})$$

Définition 1.2 Un point $p = C(t_0)$ de la courbe C s'appelle $(\nabla, \overset{1}{F}, \dots, \overset{m}{F})$ -plane, s'il existe $h_p, \overset{1}{h}_p, \dots, \overset{m}{h}_p \in R$ ainsi que :

$$(1.2) \quad (\nabla_{\dot{C}} \dot{C})(t_0) = h_p \dot{C}(t_0) + \overset{1}{h}_p \overset{1}{F}_p(\dot{C}(t_0)) + \dots + \overset{m}{h}_p \overset{m}{F}_p(\dot{C}(t_0)),$$

ou $\dot{C}(t_0) = C_* \left(\frac{d}{dt} \Big|_{t_0} \right).$

Remarque 1.3

1.3.1 Tous les points des courbes $(\nabla, \overset{1}{F}, \dots, \overset{m}{F})$ -planes sont des points $(\nabla, \overset{1}{F}, \dots, \overset{m}{F})$ -planes.

1.3.2 Si $\overset{1}{h} = \dots = \overset{m}{h} = 0$, alors (1.1) nous montre que la courbe C est ∇ -autoparallèle [17].

1.3.3 Si $\overset{1}{h} = \dots = \overset{k-1}{h} = \overset{k+1}{h} = \dots = \overset{m}{h} = 0$, alors (1.1) nous montre que la courbe C est $(\nabla, \overset{k}{F})$ -plane [7].

1.3.4 Si $\overset{1}{h} = \dots = \overset{k-1}{h} = \overset{k+1}{h} = \dots = \overset{m}{h} = 0$ et $\overset{k}{F} \circ \overset{k}{F} = -\delta$, (resp. $\overset{k}{F} \circ \overset{k}{F} = \delta$) alors (1.1) nous montre que la courbe C est $(\nabla, \overset{k}{F})$ -plane holomorphe [1], [2], [14], [16], [18], (resp. produit-plane [11]).

1.3.5 Si $\overset{1}{h} = \dots = \overset{k-2}{h} = \overset{k+1}{h} = \dots = \overset{m}{h} = 0$ et $\overset{k-1}{F} \circ \overset{k-1}{F} = -\delta$, $\overset{k}{F} \bullet \overset{k}{F} = -\delta$, $\overset{k-1}{F} \circ \overset{k}{F} + \overset{k}{F} \circ \overset{k-1}{F} = 0$ alors (1.1) nous montre que C est une courbe quaternionnien-plane [12].

1.3.6 Si $\{\delta, \overset{1}{F}, \dots, \overset{m}{F}\}$ est une base d'une algèbre commutative, associative, à élément unité, alors (1.1) nous montre que C est une H-courbe [3].

1.3.7 Soient $\overset{1}{F}_j^i, \overset{2}{F}_j^i, \dots, \overset{m}{F}_j^i$, resp. Γ_{jk}^i , les composantes de $\overset{1}{F}, \overset{2}{F}, \dots, \overset{m}{F}$, resp. ∇ , dans une voisinage de coordonnées. Alors, en utilisant (1.1) on obtient le système différentiel d'équations des courbes $(\nabla, \overset{1}{F}, \dots, \overset{m}{F})$ -planes :

$$(1.3) \quad \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = h \frac{dx^i}{dt} + \overset{11}{h} \overset{1}{F}_j^i \frac{dx^j}{dt} + \dots + \overset{mm}{h} \overset{m}{F}_j^i \frac{dx^j}{dt}.$$

1.3.8. Si l'espace engendrée par $\overset{1}{F}_j^i \frac{dx^j}{dt}, \dots, \overset{m}{F}_j^i \frac{dx^j}{dt}$ est ∇ -parallel sur la courbe C , alors C est une courbe $(\nabla, \overset{1}{F}, \dots, \overset{m}{F})$ -plane, p-géodésique speciale, où $p \leq m+1$ [4], [5].

Des champs $(\overset{1}{F}, \dots, \overset{m}{F})$ -caractéristiques

Soit $A \in \mathcal{T}_2^1(M)$. Si on définit le produit de deux champs de vecteurs $X, Y \in \mathcal{T}_0^1(M)$ par la formule $X * Y = A(X, Y)$, alors le $\mathcal{F}(M)$ -module $\mathcal{T}_0^1(M)$ devient une $\mathcal{F}(M)$ -algèbre [9]. Cette algèbre sera notée par $\mathcal{U}(M, A)$. Supposons que $A(X, Y) = A(Y, X)$, $(\forall) X, Y \in \mathcal{T}_0^1(M)$.

Définition 2.1 Soit $\overset{1}{F}, \dots, \overset{m}{F} \in \mathcal{F}_1^1(M)$. Un champ $X \in \mathcal{U}(M, A)$ s'appelle champ $(\overset{1}{F}, \dots, \overset{m}{F})$ -caractéristique s'il existe $m+1$ fonction $f, \overset{1}{f}, \dots, \overset{m}{f} \in \mathcal{F}(M)$ ainsi que :

$$(2.2) \quad A(X, X) = fX + \overset{11}{fF}(X) + \dots + \overset{mm}{fF}(X)$$

Remarque 2.2

2.2.1 Si $\overset{1}{f} = \dots = \overset{k-1}{f} = \overset{k+1}{f} = \dots = \overset{m}{f} = 0$, alors (2.2) nous montre que X est un champ $\overset{k}{F}$ -caractéristique [7].

2.2.2 Si $\overset{1}{f} = \dots = \overset{k-1}{f} = \overset{k+1}{f} = \dots = \overset{m}{f} = 0$ et $\overset{k}{F} \circ \overset{k}{F} = -\delta$, alors $X \in \mathcal{U}(M, A)$ est un champ subcaractéristique [8].

2.2.3 Si $\overset{1}{f} = \dots = \overset{m}{f} = 0$ alors X est un champ caractéristique dans l'algèbre $\mathcal{U}(M, A)$ [9], [10], [15].

2.2.4 Si $\overset{1}{f} = \dots = \overset{m}{f} = 0$, alors X est un élément nilpotent d'indice 2, dans l'algèbre $\mathcal{U}(M, A)$.

Définition 2.3 Un élément $X \in \mathcal{U}(M, A)$ s'appelle champ $(\overset{1}{F}, \dots, \overset{m}{F})$ -impropre si les vecteurs $X_p, \overset{1}{F}_p(X_p), \dots, \overset{m}{F}_p(X_p)$ sont indépendants pour tout $p \in M$.

Remarque 2.4 Dans la suite nous supposons $m = 2$.

Proposition 2.5 Soit $X \in \mathcal{U}(M, A)$ un champ $(\overset{1}{F}, \overset{2}{F})$ -impropre. Les propriétés suivantes sont équivalentes :

- (i) X est champ $(\overset{1}{F}, \overset{2}{F})$ -caractéristique.
- (ii) On a la relation :

$$(2.3) \quad \{[A(X, X) \hat{\otimes} X] \hat{\otimes} [\overset{1}{F}(X) \hat{\otimes} X]\} \hat{\otimes} \{[\overset{2}{F}(X) \hat{\otimes} X] \hat{\otimes} \hat{\otimes} [\overset{1}{F}(X) \hat{\otimes} X]\} = 0$$

$$\text{ou } T \hat{\otimes} T' = T \otimes T' - T' \otimes T.$$

Démonstration : (i) $\Rightarrow$ (ii) En utilisant la relation :

$$(2.4) \quad A(X, X) = fX + \overset{11}{fF}(X) + \overset{22}{fF}(X)$$

on obtient :

$$(2.5) \quad A(X, X) \hat{\otimes} X = \overset{1}{f}[\overset{1}{F}(X) \hat{\otimes} X] + \overset{2}{f}[\overset{2}{F}(X) \hat{\otimes} X]$$

De (2.5) on a :

$$(2.6) \quad [A(X, X) \hat{\otimes} X] \hat{\otimes} [\overset{1}{F}(X) \hat{\otimes} X] = \overset{2}{f}\{[\overset{2}{F}(X) \hat{\otimes} X] \hat{\otimes} [\overset{1}{F}(X) \hat{\otimes} X]\}$$

De (2.6) on obtient (2.3).

(ii) $\Rightarrow$ (i) De (2.3) il résulte qu'il existe une fonction $f \in \mathcal{F}(M)$ tel que on a (2.6). En utilisant (2.6) il résulte qu'il existe une fonction $f \in \mathcal{F}(M)$ tel que on a (2.5). La relation (2.5) nous montre qu'il existe une fonction $f \in \mathcal{F}(M)$ tel que on a (2.4).

Définition 2.6 Un champ $(\overset{1}{F}, \overset{2}{F})$ -caractéristique et $(\overset{1}{F}, \overset{2}{F})$ -impropre s'appelle champ de directions $(\overset{1}{F}, \overset{2}{F})$ -caractéristiques.

Définition 2.7 Les trajectoires des champs de directions $(\overset{1}{F}, \overset{2}{F})$ -caractéristique sont appelées courbes $(\overset{1}{F}, \overset{2}{F})$ -caractéristiques.

Remarque 2.8 Soit $A_{jk}^i, \overset{1}{F}_j^i$, resp. $\overset{2}{F}_j^i$, les composantes de $A, \overset{1}{F}$, resp. $\overset{2}{F}$, dans une système de coordonnées locales. Alors les relations (2.4), (2.5), (2.6), (2.3) on écrivent :

$$(2.4') \quad A_{jk}^i X^j X^k = f X^i + \overset{11}{f} \overset{1}{F}_j^i X^j + \overset{22}{f} \overset{2}{F}_j^i X^j,$$

$$(2.5') \quad (A_{jk}^i \delta_r^h - A_{jk}^h \delta_r^i) X^j X^k X^r = \overset{1}{f} (\overset{1}{F}_j^i \delta_r^h - \overset{1}{F}_j^h \delta_r^i) X^j X^r + \\ + \overset{2}{f} (\overset{2}{F}_j^i \delta_r^h - \overset{2}{F}_j^h \delta_r^i) X^j X^r,$$

$$(2.6') \quad [(A_{jk}^i \delta_r^h - A_{jk}^h \delta_r^i) (\overset{1}{F}_b^a \delta_d^c - \overset{1}{F}_b^c \delta_d^a) - (\overset{1}{F}_j^i \delta_r^h - \overset{1}{F}_j^h \delta_r^i) \cdot \\ \cdot (A_{bk}^a \delta_d^c - A_{bk}^c \delta_d^a)] X^j X^k X^r X^b X^d = \overset{2}{f} [(\overset{2}{F}_j^i \delta_r^h - \overset{2}{F}_j^h \delta_r^i) (\overset{1}{F}_b^a \delta_d^c - \\ - \overset{1}{F}_b^c \delta_d^a) - (\overset{1}{F}_j^i \delta_r^h - \overset{1}{F}_j^h \delta_r^i) (\overset{2}{F}_b^a \delta_d^c - \overset{2}{F}_b^c \delta_d^a)] X^j X^r X^b X^d,$$

$$(2.3') \quad B_{jkrb\alpha\beta\theta\sigma}^{ihac\alpha\beta\theta\sigma} X^j X^k X^r X^b X^d X^p X^q X^s X^u = 0$$

où

$$(2.3'') \quad B_{jkrb\alpha\beta\theta\sigma}^{ihac\alpha\beta\theta\sigma} = (A_{jk}^i \delta_r^h) \overset{1}{F}_b^a \delta_d^c - A_{bk}^a \delta_d^c \overset{1}{F}_j^i \delta_r^h) \cdot \\ \cdot (\overset{2}{F}_p^{\alpha} \delta_q^{\beta}) \overset{1}{F}_s^{\theta} \delta_u^{\sigma} - \overset{2}{F}_s^{\theta} \delta_u^{\sigma} \overset{1}{F}_p^{\alpha} \delta_q^{\beta}) - (A_{jk}^{\alpha} \delta_r^{\beta}) \overset{1}{F}_b^{\theta} \delta_d^{\sigma} - \\ - A_{bk}^{\theta} \delta_d^{\sigma} \overset{1}{F}_j^{\alpha} \delta_r^{\beta}) (\overset{2}{F}_p^i \delta_q^h) \overset{1}{F}_s^a \delta_u^c - \overset{2}{F}_s^a \delta_u^c \overset{1}{F}_p^i \delta_q^h).$$

Remarque 2.9 En utilisant (2.3') nous obtenons le système différentiel d'équations des courbes $(\overset{1}{F}, \overset{2}{F})$ -caractéristiques.

$$(2.3''') \quad B_{jkrb\alpha\beta\theta\sigma}^{ihac\alpha\beta\theta\sigma} \frac{dx^j}{dt} \frac{dx^k}{dt} \frac{dx^r}{dt} \frac{dx^b}{dt} \frac{dx^d}{dt} \frac{dx^p}{dt} \frac{dx^q}{dt} \frac{dx^s}{dt} \frac{dx^u}{dt} = 0.$$

Remarque 2.10 On considère de nouveau la paire (M, A) et on suppose que la variété M est dotée avec une connexion linéaire $\tilde{\nabla} \cdot A$ l'aide de A et $\tilde{\nabla}$ on construit les connexions linéaires suivantes :

$$\nabla = \tilde{\nabla} - \frac{1}{2} A, \quad \bar{\nabla} = \tilde{\nabla} + \frac{1}{2} A.$$

Proposition 2.11 Soit $C : (a, b) \rightarrow M$ une courbe régulière. Supposons que les vecteurs $\dot{C}(t)$, $\overset{1}{F}_{C(t)}(\dot{C}(t))$ et $\overset{2}{F}_{C(t)}(\dot{C}(t))$ sont indépendants pour tout $t \in (a, b)$. Les propriétés suivantes sont équivalentes :

- (i) La courbe C est $(\overset{1}{F}, \overset{2}{F})$ -caractéristique et $(\nabla, \overset{1}{F}, \overset{2}{F})$ -plane.
- (ii) La courbe C est $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -caractéristique et $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -plane.
- (iii) La courbe C est $(\nabla, \overset{1}{F}, \bar{\nabla})$ -plane et $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -plane.

Démonstration : Evidemment.

Proposition 2.12 Soit $X \in \mathcal{T}_0^1(M)$ un champ $(\overset{1}{F}, \overset{2}{F})$ -impropre. Les propriétés suivantes sont équivalentes :

- (i) X est un champ de directions $(\overset{1}{F}, \overset{2}{F})$ -caractéristiques dans l'algèbre $\mathcal{U}(M, A)$.

- (ii) Pour tout $p \in M$ il existe $f_p, \overset{1}{f}_p, \overset{2}{f}_p \in R$ tel que :

$$(2.7) \quad A_p(X_p, X_p) = f_p X_p + \overset{1}{f}_p \overset{1}{F}_p(X_p) + \overset{2}{f}_p \overset{2}{F}_p(X_p)$$

- (iii) Pour tout $p \in M$ si $C : (a, b) \rightarrow M$ est la courbe $(\nabla, \overset{1}{F}, \overset{2}{F})$ -plane qui vérifie les conditions :

$$(2.8) \quad C(t_0) = p, \quad \dot{C}(t_0) = \lambda_p X_p, \quad \lambda_p \in R - \{0\},$$

alors le point p est $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -plane.

Démonstration :

- (i) $\Rightarrow$ (ii) Puisque X est $(\overset{1}{F}, \overset{2}{F})$ -impropre il en résulte l'unicité de $f_p, \overset{1}{f}_p$ et $\overset{2}{f}_p$.

- (ii) $\Rightarrow$ (iii) Soit $C : (a, b) \rightarrow M$ la courbe $(\nabla, \overset{1}{F}, \overset{2}{F})$ -plane qui vérifie (2.8). En tenant compte de la relation (2.7) nous obtenons :

$$(2.9) \quad A_p(\dot{C}(t_0), \dot{C}(t_0)) = f_p \lambda_p \dot{C}(t_0) + \overset{1}{f}_p \lambda_p \overset{1}{F}_p(\dot{C}(t_0)) + \overset{2}{f}_p \lambda_p \overset{2}{F}_p(\dot{C}(t_0))$$

Puisque le point p est $(\nabla, \overset{1}{F}, \overset{2}{F})$ -plane il existe $a_p, \overset{1}{a}_p, \overset{2}{a}_p \in R$ tel que :

$$(2.10) \quad (\nabla \dot{C})(t_0) = a_p \dot{C}(t_0) + \overset{1}{a}_p \overset{1}{F}_p(\dot{C}(t_0)) + \overset{2}{a}_p \overset{2}{F}_p(\dot{C}(t_0))$$



En tenant compte de (2.3), (2.10) et de la relation $A = \bar{\nabla} - \nabla$ nous avons :

$$(2.11) \quad (\bar{\nabla}_c \dot{C})(t_0) = (a_p + f_p \lambda_p) \dot{C}(t_0) + (a_p + f_p \lambda_p) \overset{1}{F}_p(\dot{C}(t_0)) + \\ + (a_p + f_p \lambda_p) \overset{2}{F}_p(\dot{C}(t_0)),$$

c'est-à-dire p est un point $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -plane.

(iii) $\Rightarrow$ (ii) Evidemment, pour tout $p \in M$ il existe une courbe $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -plane $C: (a, b) \rightarrow M$ qui vérifie les conditions :

$$C(t_0) = p, \quad \dot{C}(t_0) = X_p.$$

Au plus, si le point p est $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -plane, alors nous avons des relations du type (2.10) et (2.11) où $\lambda_p = 1$ et on obtient facilement (2.7).

(ii) $\Rightarrow$ (i) Il faut montrer que les fonctions $(p \rightarrow f_p): M \rightarrow R$, $(p \rightarrow \overset{1}{f}_p): M \rightarrow R$ et $(p \rightarrow \overset{2}{f}_p): M \rightarrow R$ sont différentiables. En tenant compte de (2.7) il résulte :

$$[A_p(X_p, X_p) \hat{\otimes} X_p] \hat{\otimes} [\overset{1}{F}(X_p) \hat{\otimes} X_p] = f_p\{[\overset{2}{F}_p(X_p) \hat{\otimes} X_p] \hat{\otimes} [\overset{1}{F}_p(X_p) \hat{\otimes} X_p]\}$$

où on a notée $u \hat{\otimes} v = u \otimes v - v \otimes u$. Puisque X est $(\overset{1}{F}, \overset{2}{F})$ -impropre, il résulte :

$$[\overset{2}{F}_p(X_p) \hat{\otimes} X_p] \hat{\otimes} [\overset{1}{F}_p(X_p) \hat{\otimes} X_p] \neq 0$$

Considérons un point arbitraire $p \in M$ et soit U une voisinage de coordonnées autour de p . Il existe $i_0, h_0, a_0, c_0 \in \{1, 2, \dots, n\}$ tel que :

$$\{[\overset{2}{F}_j^{i_0}(p) \delta_r^{h_0} - \overset{2}{F}_j^{h_0}(p) \delta_r^{i_0}] [\overset{1}{F}_b^{a_0}(p) \delta_d^{c_0} - \overset{1}{F}_b^{c_0}(p) \delta_d^{a_0}] - \\ - [\overset{1}{F}_j^{i_0}(p) \delta_r^{h_0} - \overset{1}{F}_j^{h_0}(p) \delta_r^{i_0}] [\overset{2}{F}_b^{a_0}(p) \delta_d^{c_0} - \overset{2}{F}_b^{c_0}(p) \delta_d^{a_0}]\} X^j(p) X^r(p) X^b(p) X^d(p) \neq 0$$

Alors pour tout $q \in U$ nous avons :

$$\overset{2}{f}_q = \frac{\{(\overset{1}{A}_{jk}^{i_0} \delta_r^{h_0} \overset{1}{F}_b^{a_0} \delta_d^{c_0} - \overset{1}{F}_j^{i_0} \delta_r^{h_0} \overset{1}{A}_{bk}^{a_0} \delta_d^{c_0})\} X^j X^k X^r X^b X^d(q)}{\{(\overset{2}{F}_l^{i_0} \delta_s^{h_0} \overset{1}{F}_u^{a_0} \delta_t^{c_0} - \overset{2}{F}_l^{i_0} \delta_s^{h_0} \overset{1}{F}_u^{a_0} \delta_t^{c_0})\} X^l X^s X^u X^t(q)}$$

donc la fonction $(q \rightarrow \overset{2}{f}_q)$ est différentiable.

De la relation (2.5) on a :

$$(2.5^{IV}) \quad A_p(X_p, X_p) \hat{\otimes} X_p = \overset{1}{f}_p[\overset{1}{F}_p(X_p) \hat{\otimes} X_p] + \overset{2}{f}_p[\overset{2}{F}_p(X_p) \hat{\otimes} X_p]$$

Soit $p \in M$. Puisque $F_p(X_p) \hat{\otimes} X_p \neq 0$, il résulte qu'il existe un voisinage de coordonnées V autour de p et $i_0, h_0 \in \{1, \dots, n\}$ tel que la fonction :

$$(F_j^{i_0} \delta_r^{h_0} - F_j^{h_0} \delta_r^{i_0}) X^j X^r : V \rightarrow R$$

ne s'annule pas en aucun point.

Pour tout $q \in V$ on a :

$$f_q = \frac{A_{jk}^{i_0}(q) \delta_r^{h_0} X^j(q) X^k(q) X^r(q) - f_q^2 F_j^{i_0} \delta_r^{h_0} X^j(q) X^r(q)}{F_l^{i_0}(q) \delta_s^{h_0} X^l(q) X^s(q)}$$

Donc la fonction $(q \rightarrow f_q)$ est différentiable.

Soit $p \in M$. Puisque $X_p \neq 0$ il résulte qu'il existe un voisinage de coordonnées W autour de p et $i_0 \in \{1, 2, \dots, n\}$ tel que $X^{i_0}(q) \neq 0$, $(\forall) q \in W$. Alors de (2.7) on obtient :

$$f_q = \frac{1}{X^{i_0}(q)} \{A_{jk}^{i_0}(q) X^j(q) X^k(q) - f_q^1 F_j^{i_0}(q) X^j(q) - f_q^2 F_j^{i_0}(q) X^j(q)\}$$

donc la fonction $(q \rightarrow f_q)$ est différentiable.

§ 3. Applications

I) Soit M une hypersurface dans l'espace euclidien E^{n+1} , où $n \geq 2$. Soient g_{ij} , resp. b_{ij} , les composantes de la première, resp. la deuxième forme fondamentale. Supposons que $\det \|b_{ij}\| \neq 0$. Soient Γ_{jk}^i et $\bar{\Gamma}_{jk}^i$ les composantes des connexions linéaires associées à g_{ij} et b_{ij} . Soit $A \in \mathcal{T}_2^1(M)$ ayant les composantes

$$(3.1) \quad A_{jk}^i = \bar{\Gamma}_{jk}^i - \Gamma_{jk}^i$$

Nous avons les formules [7], [13] :

$$(3.2) \quad A_{jk}^i = \frac{1}{2} b^{ij} b_{jk,s}$$

où $b_{jk,s}$ sont les dérivées covariantes du tenseur b_{jk} par rapport à l'aide de la connexion Γ_{jk}^i et où b^{ij} sont données par :

$$(3.3) \quad b^{ij} b_{jk} = \delta_k^i$$

Supposons que tous les points de M sont des points nonombilicals.

Soient $F^1, F^2 \in \mathcal{T}_1^1(M)$, définies par

$$(3.4) \quad F_j^1 = g^{jk} b_{kj}, \quad F_j^2 = b^{jk} g_{kj}$$

où F_j^1 , resp. F_j^2 sont les composantes de F^1 , resp. F^2 .

Définition 3.1 Les champs $(\overset{1}{F}, \overset{2}{F})$ -caractéristiques de l'algèbre $\mathcal{U}(M, A)$ sont appelées champs $(\overset{1}{F}, \overset{2}{F})$ -caractéristiques de l'hypersurface M .

Remarque 3.2 En utilisant (2.2) on obtient que un champ $X \in \mathcal{T}_0^1(M)$ est de champ $(\overset{1}{F}, \overset{2}{F})$ -caractéristique de l'hypersurface M s'il existe $\overset{1}{f}, \overset{2}{f}$, $f \in \mathcal{T}(M)$ tel que :

$$(3.5) \quad b_{jk,i} X^j X^k = f b_{ik} X^k + \overset{1}{f} g^{sk} b_{si} b_{kj} X^j + \overset{2}{f} g_{ij} X^j$$

Supposons que $\overset{1}{f} = \overset{2}{f}$. On a :

Proposition 3.2 [7]. Tout élément $X \in \mathcal{U}(M, A)$ est un champ $(\overset{1}{F} + \overset{2}{F})$ -caractéristique si et seulement si M est sphère.

II) Soit M une variété différentiable à n dimensions ($n > 3$) et soient $\overset{1}{F}, \overset{2}{F} \in \mathcal{T}_1^1(M)$. Supposons que :

$$\overset{1}{F} \circ \overset{1}{F} = \delta, \quad \overset{2}{F} \circ \overset{2}{F} = \delta, \quad \overset{1}{F} \circ \overset{2}{F} = \overset{2}{F} \circ \overset{1}{F} = \overset{1}{F} - \overset{2}{e} \overset{2}{F} + e \delta, \quad e = \pm 1,$$

Soient $\overset{1}{N}, \overset{2}{N} \in \mathcal{T}_2^1(M)$ les champs tensoriel de Nijenhuis données par :

$$\begin{aligned} \overset{i}{N}(X, Y) &= [\overset{i}{F}(X), \overset{i}{F}(Y)] - \overset{i}{F}([\overset{i}{F}(X), Y]) - \overset{i}{F}([X, \overset{i}{F}(Y)]) + \\ &+ \overset{i}{F} \circ \overset{i}{F}([X, Y]), \end{aligned}$$

ou $i = 1, 2$. Nous avons [6].

Lemme 3.6 L'espace fibré TM est de la forme

$$TM = \overset{1}{F}^{(e)} \oplus \overset{2}{F}^{(-1)} \oplus \overset{1}{F}^{(e)} \cap \overset{2}{F}^{(1)},$$

ou $\overset{i}{F}^{(1)}, \overset{i}{F}^{(-1)}$ sont des distributions sur la variété M et $\overset{i}{F}(X) = eX$, $e = \pm 1$.

Lemme 3.7 S'il existe sur la variété M , une connexion linéaire symétrique ∇ telle que :

$$(3.6) \quad \nabla_X \overset{i}{F}(X) = \overset{\omega}{1} \overset{i}{\omega}(X) \overset{1}{F}(Y) + \overset{\omega}{2} \overset{i}{\omega}(X) \overset{2}{F}(Y) + \overset{\omega}{0} \overset{i}{\omega}(X) Y + \frac{1}{2} \overset{i}{F}(\overset{i}{N}(X, Y)),$$

ou $X, Y \in \mathcal{T}_0^1(M)$, $\overset{\omega}{0}, \overset{\omega}{2}, \overset{\omega}{1} \in \mathcal{T}_1^0(M)$, alors ont lieu les relations :

$$(3.7) \quad \overset{i}{N}(X, Y) + \overset{i}{F}(\overset{i}{N}(\overset{i}{F}(X), Y)) + \overset{2}{e} \overset{2}{F}(\overset{2}{N}(X, Y)) - \overset{1}{F}(\overset{1}{N}(X, Y)) = 0$$

$$\text{ou } k = \begin{cases} 1 & \text{si } i = 2, \\ 2 & \text{si } i = 1, \end{cases}$$

$$(3.8.1) \quad \begin{cases} \omega_0^1(X) = -\omega_1^1(\overset{1}{F}(X)) = -\omega(\overset{1}{F}(X)), \\ \omega_0^2(X) = -\omega_2^2(\overset{2}{F}(X)) = -\omega(\overset{2}{F}(X)) - \varepsilon(X), \\ \varepsilon(\overset{2}{F}(X)) = -\varepsilon(\overset{1}{F}(X)) = \varepsilon \end{cases}$$

$$(3.8.2) \quad \begin{cases} \omega_2^1 = \omega_1^2 = 0, \quad \omega_1^1 = \omega_2^2 = \omega, \quad \text{pour } e = -1, \\ \omega_1^1 = \omega, \quad \omega_2^2 = \omega + \varepsilon, \quad \text{pour } e = 1, \end{cases}$$

ou $\omega, \varepsilon \in \mathcal{T}_1^0(M)$.

Théorème 3.8 Supposons qu'on ait (3.7) et (3.8). Alors, il existe une connexion linéaire symétrique ∇ sur M qui vérifie les conditions (3.6).

Corollaire 3.9 Si

$$A(X, Y) = \omega_0(X)Y + \omega_0(Y)X + \omega_1(X)\overset{1}{F}(Y) + \omega_1(Y)\overset{1}{F}(X) + \\ + \omega_2(X)\overset{2}{F}(Y) + \omega_2(Y)\overset{2}{F}(X), \quad \omega_0, \omega_1, \omega_2 \in \mathcal{T}_1^0(M)$$

pour ∇ la connexion linéaire symétrique vérifiant (3.6), alors le difféomorphisme

$$\pi_1(3) : (M, \bar{\nabla}) \rightarrow (M, \nabla),$$

ou

$$\nabla = \bar{\nabla} + A,$$

est un difféomorphisme 3-géodésique linéaire du premier type [6].

3.9.1 Si $C : (a, b) \rightarrow (M, \bar{\nabla})$ est une courbe géodésique alors $\pi_1(3) \circ C$ est une courbe avec les trois propriétés suivantes :

- a) une courbe $(\nabla, \overset{1}{F}, \overset{2}{F})$ -plane.
- b) une courbe 3-géodésique.
- c) une courbe hypercomplexe.

3.9.2 Si $C : (a, b) \rightarrow (M, \bar{\nabla})$ est une courbe géodésique et

- a) $\dot{C}(t) \in \overset{i}{F}^{(e)}$,

ou

- b) $\dot{C}(t) \in \overset{i}{F}^{(1)} \cap \overset{h}{F}^{(-1)}$,

alors $\pi_1(3) \circ C(t)$ est une courbe $(\nabla, \overset{i}{F})$ -plane respectivement courbe géodésique.

3.9.3 Il en résulte que les courbes $(\nabla, \overset{1}{F}, \overset{2}{F})$ -planes et $(\bar{\nabla}, \overset{1}{F}, \overset{2}{F})$ -planes sont les mêmes.

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ON THE COMPENSATED INACCESSIBLE JUMPS OF A SQUARE INTEGRABLE MARTINGALE

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Résumé

Dans ce travail on fait quelques remarques sur l'aspect du compensé d'un saut inaccessible d'une martingale à carré intégrable, remarques qui sont du, en essence, au fait qu'il existe un plus grand temps d'arrêt qui soit strictement plus petit qu'un temps d'arrêt totalement inaccessible, sur l'ensemble où ce dernier est fini.

It is well known that any square integrable martingale, may be decomposed into a continuous martingale and a purely discontinuous one. The purely discontinuous parts is "the compensated sum of its jumps". If the jump is along a predictable stopping time, then "the compensator" is zero. If the jump is along a totally inaccessible stopping time, then "the compensator" is a continuous process with integrable variation. This paper is intended to do some remarques about the aspect of this process.

We shall use the notation, the terminology and certain results of the theory of martingales as they are exposed in [1], [2] or [3].

We place ourselves in the following context. Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space and $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be an increasing family of sub- σ -algebras of $\mathcal{F}$ which is right continuous and such that $\mathcal{F}_0 \supset \{\Lambda \mid \Lambda \in \mathcal{F}, P(\Lambda) = 0\}$. We suppose also that $\mathcal{F}$ is generated by $\bigcup_{t \in \mathbb{R}_+} \mathcal{F}_t$.

We shall not distinguish in notations between a class of random variables which are equally a.s. and its representatives. Also, we shall identify indistinguishable processes, i.e. processes which differ only on a set of trajectories of probability zero.

In the following proposition we prove that, there exists a greatest stopping time, strictly smaller than a totally inaccessible stopping time, on the set where this totally inaccessible stopping time is finite.

Proposition. Let T be a totally inaccessible stopping time. There then exists a stopping time S , so that

- i) $\{T < \infty\} \subset \{S < T\}$ a.s.
- ii) if U denotes a stopping time satisfying

$$\{T < \infty\} \subset \{U < T\},$$

then $U \leq S$ a.s.

Two such stopping times differ on a set of probability zero.

Proof: Let

$$\mathcal{S} = \{U \mid U \text{ stopping time and } \{T < \infty\} \subset \{U < T\}\}$$

The set $\mathcal{S}$ is not empty: $0 \in \mathcal{S}$, since $P\{T = 0\} = 0$, T being totally inaccessible. Also we remark that $\mathcal{S}$ is hereditary: if $U \in \mathcal{S}$ and V is a stopping time satisfying $V \leq U$, then $V \in \mathcal{S}$. It results that $\mathcal{S}$ is stable under countable infimum.

Let us prove that $\mathcal{S}$ is in fact, a complete σ -lattice. If $U, V \in \mathcal{S}$, then $\sup(U, V) \in \mathcal{S}$, since

$$\{T < \infty\} \subset \{U < T\} \cap \{V < T\} \subset \{\max(U, V) < T\}.$$

Let now (U_n) be a sequence of elements of $\mathcal{S}$. Since $\mathcal{S}$ is a lattice, we may suppose that this sequence increases. We have that, for any $n \in N$,

$$\{T < \infty\} \subset \{U_n < T\} \text{ a.s.}$$

or, equivalently

$$\{T < \infty\} \subset \{U_n < T < \infty\} \text{ a.s.,}$$

hence

$$\{T < \infty\} \subset \bigcap_n \{U_n < T < \infty\} \text{ a.s.}$$

If we denote $U = \sup_n U_n$, then we have

$$\begin{aligned} \bigcap_n \{U_n = T < \infty\} &= \{U = T < \infty, U_n < T \text{ for any } n \in N\} \cup \\ &\cup \{U < T < \infty, U_n < T \text{ for any } n \in N\}; \end{aligned}$$

but

$$P\{U = T < \infty, U_n < T \text{ for any } n \in N\} = 0$$

since T is totally inaccessible. Hence,

$$\bigcap_n \{U_n < T < \infty\} = \{U < T < \infty, U_n < T \text{ for any } n \in N\} \text{ a.s.}$$

which implies that

$$\{T < \infty\} \subset \{U < T\} \text{ a.s.}$$

and $U \in \mathcal{S}$.

Let us choose now

$$S = \sup_{\mathcal{S}} S,$$

i.e. S satisfies: $S \geq U$, for any $U \in \mathcal{S}$ a.s., and if V is random variable such that $V \geq U$ a.s., for any $U \in \mathcal{S}$, then $V \geq S$ a.s. If we show that $S \in \mathcal{S}$, then the proof will be complete. Indeed, S may be considered, for instance, as the limit of an increasing sequence (S_n) of elements form $\mathcal{S}$, such that

$$\lim_{n \rightarrow \infty} \int \frac{S_n}{1 + S_n} dP = \sup_{U \in \mathcal{S}} \int \frac{U}{1 + U} dP.$$

Hence S is a stopping time and $S \in \mathcal{S}$, since $\mathcal{S}$ is stable under countable supremum.

The assertion concerning the unicity of S is trivial.

If $f \in L_2(\Omega, \mathcal{F}, P)$, we choose $m^f = (m_t^f)_{t \in R_+}$ to stand for a process whose trajectories, are right continuous, possessing finite limites at the left and which satisfies, for any $t \in R_+$

$$m_t^f = E(f | \mathcal{F}_t) \quad \text{a.s.}$$

We remind of a result very often used in this theory. If m is a continuous martingale, null at the moment 0 and with integrable variation then m is null. Indeed, if we denote by Pm the measure on $(\Omega \times R_+, \mathcal{F} \otimes \mathcal{B}_+)$, defined as

$$Pm(x) = \iint x_t dm_t dP,$$

where x is a bounded measurable process, then we have

$$Pm = Pm \circ \pi^p = 0,$$

where π^p is the previsible projection.

Theorem: Let T be a totally inaccessible stopping time and $f \in L_2(\Omega, \mathcal{F}, P)$ be such that m^f is purely discontinuous and continuous outside the graph of T^* . Then, if S denotes the greatest stopping time strictly smaller than T , where T is finite, we have that:

- a) m^f is null on the stochastic interval $[0, S]$
- b) m^f is stationary on the stochastic interval $[T, \infty)$

Proof.: Let us consider the martingale $m^{f,S} = (m_{\min(t,S)}^f)_t$. One may easily remark that this martingale is continuous, null at zero and with integrable variation. It follows that $m^{f,S} = 0$, hence m^f is zero on $[0, S]$.

Let us consider now the martingale $m^{f,T} = (m_{\min(t,T)}^f)_t$. The martingale $m^f - m^{f,T}$ is continuous since, at the moment T , m^f and $m^{f,T}$ have same jumps. Also, this martingale has integrable variation and is null at the moment zero, hence $m^f - m^{f,T} = 0$. In this way, we get that

* [3], ch. II, no 8, 9: If $g = m_T^f - m_{T-}^f$, then m^f is the compensated of the process $A_t = g \chi_{[T, \infty)}$ and the compensator of A is a continuous process with integrable variation.

$m^f = m^{f,T}$ on the stochastic interval $[T, \infty)$ from which it results that m^f is stationary on $[T, \infty)$.

The theorem is proved.

Corollary: Let T be a totally inaccessible stopping time and S be the greatest stopping time smaller than T where T is finite. If $g \in L_2(\Omega, \mathcal{F}_T, P)$, then the compensator of $A = g\chi_{[T, \infty)}$ is null on $[0, S]$ and stationary on $[T, \infty)$.

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To memory of Professor Gr. C. Moisil
(at the 75th anniversary)

NO TOURNAMENT IS GRADABLE

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A *digraph* [1] (directed graph) D consists of a set $V(D)$ of *vertices* and a set $E(D)$ of ordered pairs xy of distinct vertices called *edges*. A *path* [1] is a digraph with vertex set $\{x_1, x_2, \dots, x_n\}$ and edge set $\{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\}$. This path is called an x_1x_n path and is denoted $p(x_1, x_n)$. We denote by $l[p(x_1, x_n)]$ the *length* [1] of an x_1x_n path, i.e., its number of edges. A *circuit* [1] $(x_1, x_2, \dots, x_n, x_1)$ is the digraph obtained from the path x_1x_n by adding the edge x_nx_1 . A digraph without circuits is *gradable* [3] if there exist the non-negative integers $m(x_1), m(x_2), \dots, m(x_n)$, associated to the vertex $x_1, x_2, \dots, x_n$, such that for every edge xy holds $m(y) - m(x) = 1$. Note that a digraph with circuits cannot be gradable. A *tournament* [2] is a digraph such that each pair of vertices is joined by precisely one edge. For an arbitrary vertex x of a digraph we denote $V(x) = \{y \in V(D) \mid \text{there exists an } xy \text{ path}\} \cup \{x\}$.

Lemma. Let D be a digraph without circuits such that there exists an unique $x_0 \in V(D)$ with $V(x_0) = V(D)$. D is gradable if and only if for every two paths $p_1(x, y)$ and $p_2(x, y)$ holds $l[p_1(x, y)] = l[p_2(x, y)]$, for all $x, y \in V(D)$.

Proof: Obviously, if D is gradable, for every path $p(x, y)$ we have $l[p(x, y)] = m(y) - m(x)$. Hence $l[p_1(x, y)] = l[p_2(x, y)] = m(y) - m(x)$.

We attach to x_0 an arbitrary non-negative integer $m(x_0)$ and to every $y \in V(D)$, $y \neq x_0$, the number $m(y) = m(x_0) + l(x_0, y)$, where $l(x_0, y)$ is the length of a path from x_0 to y , (all these paths have the same length), and let xy be an arbitrary edge of D . Since D is without circuits and x is reachable from x_0 it is impossible to have $x \neq x_0$ and $y = x_0$. If $x = x_0$ and $y \neq x_0$ we have: $m(y) - m(x) = m(x_0) + l(x_0, y) - m(x_0) = l(x_0, y) = 1$. If $x \neq x_0$ and $y \neq x_0$ we have: $m(y) - m(x) = m(x_0) + l(x_0, y) - m(x_0) - l(x_0, x) = l(x_0, y) - l(x_0, x)$. But $l(x_0, y) = l(x_0, x) + 1$. Hence $m(y) - m(x) = 1$, and D is gradable. The vertex x_0 is unique because D does not contain circuits.

Theorem: There is no gradable tournament.

Proof: In a tournament between every pair of vertex x, y there exists a path of length 1. (the edge xy or yx). But every other path between x and y or y and x , is of a length > 1 . According to lemma it follows that no tournament can be gradable.

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A METHOD FOR FINDING THE OFFERED TELEPHONE TRAFFIC FROM FLOWED TELEPHONE TRAFFIC ACCORDING TO ERLANG'S EQUATION

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In *telephone traffic studies* [2] related to growth and optimizations in town and trunk telephone networks must be find the *offered traffic* (Y) from *flowed traffic* (X) according to Erlang's equation [2]

$$X = Y(1 - P), \quad (1)$$

where

$$(2) \quad P(N, Y) = \frac{Y^N}{N! \sum_{i=0}^N \frac{Y^i}{i!}}$$

is Erlang's *probability of loss* [2] and N is the *number of circuits* for which was measured X .

The aim of this paper is to suggest a method for solve the equation (1), using *contraction principle* [1].

Theorem 1. Let $[a, b] \subset \mathbb{R}$ be and $f: [a, b] \rightarrow [a, b]$ such that there exists $\lambda \in \mathbb{R}$, $\lambda < 1$ for which $|f(x) - f(y)| \leq \lambda |x - y|$, $\forall x, y \in [a, b]$. The followings hold

(3) f is continue on $[a, b]$,

(4) for every $x_0 \in [a, b]$ the row $x_0, x_1, x_2, \dots, x_n, \dots$, defined by $x_{n+1} = f(x_n)$ converges with $\lim_{n \rightarrow \infty} x_n = x^*$, $f(x^*) = x^*$ and $|x_n - x^*| \leq \frac{\lambda^n}{1 - \lambda} |x_0 - x_1|$,

(5) there exists an unique $\tilde{x} \in [a, b]$ such that $f(\tilde{x}) = \tilde{x}$.

Proof. (3). Let $\{x_n\}_{n \in \mathbb{N}}$ be a row with $\lim_{n \rightarrow \infty} x_n = x_0 \in [a, b]$. We have

$\lim_{n \rightarrow \infty} |f(x_n) - f(x_0)| \leq \lim_{n \rightarrow \infty} \lambda |x_n - x_0| = \lambda \lim_{n \rightarrow \infty} |x_n - x_0| = 0$, i.e., f is continue. (Q.E.D.)

(4). Obviously hold

$$|x_n - x_{n+p}| \leq \sum_{i=0}^{p-1} |x_{n+i} - x_{n+i+1}|,$$

$$|x_k - x_{k+1}| = |f(x_{k-1}) - f(x_k)| \leq \lambda |x_{k-1} - x_k| \leq \dots \leq \lambda^k |x_0 - x_1|$$

and lead to

$$|x_n - x_{n+p}| \leq \sum_{i=0}^{p-1} \lambda^{n+i} |x_0 - x_1| \leq \frac{\lambda^n}{1-\lambda} |x_0 - x_1|.$$

Hence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy row, i.e., converges. Then there exists $x^* \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Because f is continue and $x_{n+1} = f(x_n)$ it follows

$$x^* = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n) = f(x^*). \quad (\text{Q.E.D.})$$

Also we have

$$|x_n - x^*| = \lim_{p \rightarrow \infty} |x_n - x_{n+p}| \leq \frac{\lambda^n}{1-\lambda} |x_0 - x_1|. \quad (\text{Q.E.D.})$$

(5). Suppose there exist $\tilde{x}_1, \tilde{x}_2 \in [a, b]$ such that $f(\tilde{x}_1) = x_1$ and $f(\tilde{x}_2) = \tilde{x}_2$. Then we have

$$|\tilde{x}_1 - \tilde{x}_2| = \lambda |f(\tilde{x}_1) - f(\tilde{x}_2)| \leq \lambda |\tilde{x}_1 - \tilde{x}_2|,$$

contradiction with hypothesis ($\lambda < 1$). (Q.E.D.)

Theorem 2.: Let $x_0 \in \mathbb{R}$, $r \in \mathbb{R}$, $S(x_0, r) = \{y \in \mathbb{R}, |x - x_0| \leq r\}$ be and $f: S(x_0, r) \rightarrow \mathbb{R}$ such that

(6) there exists $\lambda \in \mathbb{R}$, $\lambda < 1$ with $|f(x) - f(y)| \leq \lambda |x - y|$, for all $x, y \in S(x_0, r)$,

(7) $|x_0 - f(x_0)| \leq r(1 - \lambda)$.

In these conditions holds $f[S(x_0, r)] \subseteq S(x_0, r)$.

Proof.: Let $x \in S(x_0, r)$ be. Holds

$$\begin{aligned} |f(x) - x_0| &\leq |f(x) - f(x_0)| + |f(x_0) - x_0| \leq \lambda |x - x_0| + \\ &+ r(1 - \lambda) \leq \lambda r + r(1 - \lambda) = r. \end{aligned}$$

Hence $f(x) \in S(x_0, r)$. (Q.E.D.)

Theorem 3. Let $f: [a, b] \rightarrow \mathbb{R}$ be, continue on $[a, b]$ and derivable on (a, b) such that that $\sup_{x \in (a, b)} |f'(x)| < 1$. If $\left| f\left(\frac{b+a}{2}\right) - \frac{b+a}{2} \right| \leq \frac{b-a}{2} (1-\lambda)$, where $\lambda = \sup_{x \in (a, b)} |f'(x)|$, then there exists an unique $x^* \in [a, b]$ with $f(x^*) = x^*$.

Proof.: According to Lagrange's theorem [3] holds

$$|f(x) - f(y)| = |x - y| |f'(\xi)| \leq \lambda |x - y|, \text{ with } \xi \in (a, b).$$

According to theorem 1 and theorem 2, for $x_0 := \frac{a+b}{2}$ and $r := \frac{b-a}{2}$, it follows the conclusion of theorem 3. (Q.E.D.)

Corollary: In theorem 3's conditions. choosing arbitrary $x_1 \in [a, b]$ and defining the row $x_1, x_2, \dots, x_n, \dots$, by $x_{n+1} = f(x_n)$, then $\lim_{n \rightarrow \infty} x_n = x^*$ and

$$|x_n - x^*| \leq \frac{\lambda^{n-1}}{1 - \lambda} |x_2 - x_1|.$$

Proof.: Evidently according to theorems 1, 2 and 3. (Q.E.D.)
We denote

$$S_N(y) = 1 + \sum_{i=1}^N \frac{y^i}{i!} \text{ and } f(y) = X + \frac{y^{N+1}}{N! S_N(y)}, \quad y \in \mathbb{R}.$$

Theorem 4. For every $M > X$ holds

$$\sup_{y \in (X, M)} |f'(y)| < 1.$$

Proof.: Obviously $S'_N(y) = S_{N-1}(y)$, where $S'_N(y)$ is the derivate of $S_N(y)$. We have

$$f'(y) = \frac{y^N [(N+1) S_N(y) - y S_{N-1}(y)]}{N! [S_N(y)]^2}.$$

It is obviously to see that

$$(N+1) S_N(y) - y S_{N-1}(y) = \sum_{i=0}^N S_i(y) > 0.$$

Hence

$$|f'(y)| = \frac{y^N \sum_{i=0}^N S_i(y)}{N! [S_N(y)]^2}$$

We must show that

$$y^N \sum_{i=0}^N S_i(y) < N! [S_N(y)]^2$$

We prove by induction on N . For $N = 1$ it is evidently.
Suppose that is true for N . Evidently

$$(N+1) S_{N+1}(y) - y S_N = \sum_{i=0}^N S_i(y) > 0,$$

or

$$(N+1) S_{N+1}(y) > y S_N(y).$$

This last inequality is equivalent to

$$(N+1)! S_{N+1}(y) S_N(y) > yN! [S_N(y)]^2.$$

By induction hypothesis it follows

$$(N+1)! S_{N+1}(y) S_N(y) > yN! [S_N(y)]^2 > y^{N+1} \sum_{s=0}^N S_s(y).$$

Hence

$$(N+1)! S_{N+1}(y) S_N(y) > y^{N+1} \sum_{i=0}^N S_i(y).$$

Adding in this inequality $y^{N+1} S_{N+1}(y)$ we obtain

$$(N+1)! S_{N+1}(y) \left[S_N(y) + \frac{y^{N+1}}{(N+1)!} \right] > y^{N+1} \sum_{i=0}^N S_i(y) + y^{N+1} S_{N+1}(y),$$

or

$$y^{N+1} \sum_{i=0}^N S_i(y) < (N+1)! [S_{N+1}(y)]^2. \quad (\text{Q.E.D.})$$

Remark : For $M^* \in \mathbb{R}$ with $X < M^*$ and $M \in \mathbb{R}$ with

$$(8) \quad X \leq M \frac{1-\lambda}{3-\lambda},$$

where $\lambda = \sup_{y \in (X, M^*)} |f'(y)|$, we have, according to theorem 4,

$$(9) \quad \lambda < 1.$$

From (8) and (9) it follows

$$X(3-\lambda) \leq M(1-\lambda)$$

or

$$X + 2X - X\lambda = 3X - X\lambda \leq M - M\lambda$$

or

$$X + M + 2X - X\lambda \leq 2M - M\lambda$$

or

$$M + X \leq 2(M - X) + \lambda(X - M) = 2(M - X) - \lambda(M - X)$$

or

$$M + X \leq (M - X)(2 - \lambda)$$

or, because $0 < P < 1$,

$$P(M + X) \leq (M - X)(2 - \lambda) = M - X + (M - X)(1 - \lambda),$$

or

$$\frac{X - M + P(M + X)}{2} \leq \frac{(M - X)(1 - \lambda)}{2}$$

or

$$\frac{X + XP + MP - M}{2} \leq \frac{(M - X)(1 - \lambda)}{2}$$

or

$$\frac{2X + XP + MP - M - X}{2} \leq \frac{(M - X)(1 - \lambda)}{2}$$

or

$$(10) \quad \left| f\left(\frac{M + X}{2}\right) - \frac{M + X}{2} \right| = \left| X + \frac{X + M}{2}P - \frac{M + X}{2} \right| \leq \\ \leq \left| \frac{(M - X)(1 - \lambda)}{2} \right| = \frac{M - X}{2}(1 - \lambda).$$

According to (10) f defined in theorem 4 verifies the conditions of theorem 3 on $[X, M]$. Hence there exists unique $Y^* \in [X, M]$ such that $f(Y^*) = Y^*$ or

$$(11) \quad Y^* = X + Y^*P,$$

and Y^* can be obtained according to corollary.

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We mention that for the calculus of Y^* we have written a FORTRAN programme (according to the method suggested in this paper). This programme compute Y^* starting from X after three iterations. In this way we proved the efficiency of the suggested method.

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$$\frac{1 - V + VM + X}{2} \geq \frac{(1 - V)(1 - X)}{2}$$

$$\frac{1 - 2V + VM - X}{2} \geq \frac{(1 - V)(1 - X)}{2}$$

$$\frac{1 - 2V + VM - X}{2} \geq \frac{(1 - V)(1 - X)}{2}$$

$$\left(\frac{1 - 2V + VM - X}{2} \right) \geq \left(\frac{(1 - V)(1 - X)}{2} \right)$$

$$\frac{(1 - 2V + VM - X)}{2} \geq \frac{(1 - V)(1 - X)}{2}$$

According to (10), defined in theorem 4, verify the condition of theorem 3 on V . Hence, there exists unique $V^* \in [V, M]$ such that $V^* =$

$$V^* = \frac{1 - 2V + VM - X}{1 - V}$$

and V^* can be obtained according to corollary.

We mention that for the values of V^* we have written a FORTRAN program according to the method suggested in this paper. This program computes V^* starting from V after three iterations. In this way we proved the efficiency of the suggested method.

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ON THE APPLICATION OF THE PRINCIPLE OF MAXIMUM INDETERMINATION TO THE FINITE DISCRETE DISTRIBUTION

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It is known, being given a random variable f which takes a finite number of real values $f_1 < \dots < f_n$ when its mean value $E(f)$ is known, the corresponding discrete distribution is not uniquely determined. More exactly, if $n > 2$ there is an endless number of discrete distributions of the type.

$$P_k > 0, \quad \sum_{k=1}^n P_k = 1$$

compatible with the number $E(f)$, i.e. an infinity of distributions which solve the equality.

$$E(f) = \sum_{k=1}^n f_k P_k$$

Bringing in as a selection criterium the maximization of the entropy

$$H_n = - \sum_{k=1}^n P_k \ln P_k$$

the discrete distribution of maximum indetermination compatible with the mean value $E(f)$ is

$$P_k = \frac{1}{\Phi(\beta)} e^{-\beta f_k}$$

where

$$\Phi(\beta) = \sum_{k=1}^n e^{-\beta f_k}$$

β being the unique solution of the equation.

$$(1) \quad \frac{1}{\Phi(\beta)} \frac{d\Phi(\beta)}{d\beta} = -E(f)$$

Although in many other situations the application of the maximum entropy principle gives exact solution expressible by means of elementary functions, the particular finite case showed above does not allow us to obtain an exact value for β solution of the equation mentioned, this solution being transcendental.

Consequently, it is necessary to solve this equation approximatively. We present herewith a method to get an approximative solution β

The equation (1) can be written :

$$\frac{f_1 e^{-f_1 \beta} + \dots + f_n e^{-f_n \beta}}{e^{-f_1 \beta} + \dots + e^{-f_n \beta}} = E(f)$$

$$- (f_1 - E(f)) e^{-(f_1 - E(f)) \beta} - \dots - (f_n - E(f)) e^{-(f_n - E(f)) \beta} = 0$$

or :

$$a_1 e^{-a_1 \beta} + \dots + a_n e^{-a_n \beta} = 0 \quad \text{where}$$

$$a_i = - (f_i - E(f)), \quad (i = 1, 2, \dots, n)$$

and because $f_1 < \dots < f_n$ then there is $i_0, 1 \leq i_0 \leq n-1$ so that $a_1 > \dots > a_{i_0} > 0 > a_{i_0+1} > \dots > a_n$

To solve the last equation we shall take the function

$$F(x) = e^{a_1 x} + \dots + e^{a_n x}$$

We notice that F is twice derivable and because

$$F''(x) = a_1^2 e^{a_1 x} + \dots + a_n^2 e^{a_n x}$$

it is also strictly convex.

So, all conditions are satisfied to say that solving of the equation $F'(x) = 0$ is equivalent with the minimisation of function F , i.e. the unique minimum value of function F is a solution to the equation $F'(x) = 0$

We shall find out the solution of the equation $F'(x) = 0$ applying a minimization algorithm to the function F , namely the Newton method with variable step.

We chose this method mainly because it does not need an initial good enough approximation of the solution, the convergence being satisfied whatever the initial iteration x_0 might be.

The sequence of iterations is built finding out an X_{k+1} of the type

$$x_{k+1} = x_k - \alpha_k \frac{F'(x_k)}{F''(x_k)} \quad \text{where } \alpha_k \text{ is calculated as follows :}$$

$$1. \alpha = 1$$

$$2. x = x_k - \alpha \frac{F'(x_k)}{F''(x_{k+1})}$$

3. then calculate $F(x)$

4. then check out $F(x) - F(x_k) \leq -\varepsilon \alpha \frac{[F'(x_k)]^2}{F''(x_k)}$, $\left(0 < \varepsilon < \frac{1}{2}\right)$

5. If the inequality is satisfied, make $\alpha_k = \alpha$; if not, replace α by $\frac{\alpha}{2}$ and go on with point no. 2.

The interpretation of β as a minimum value of function F allows us to get apriori an interval to which belongs the solution.

We notice that:

$$F\left(\frac{\ln n}{a_1}\right) > n, \quad F\left(\frac{\ln n}{a_n}\right) > n \text{ and } n = F(0) \geq \min F$$

therefore solution β belongs to the interval $\left[\frac{\ln n}{a_n}, \frac{\ln n}{a_1}\right]$

This remark is very important holding in view the implementation of the algorithm described on the computer.

Remark. This algorithm was tested on FELIX C-256 SYSTEM and for:

$$n = 6, \quad E(f) = 10 \text{ and } f_i = 2^i, \quad (i = 1, 2, \dots, 6)$$

we obtained $\beta = 0.0389671$ in 4 seconds

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DES CHAMPS SPÉCIAUX DANS L'ALGÈBRE DE DÉFORMATION

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Dans cette Note on introduit les champs spéciaux dans l'algèbre de déformation de deux connexions linéaires sur une variété différentiable M . On met puis en évidence les courbes spéciales de l'algèbre de déformation et on étudie les courbes spéciales d'une hypersurface dans l'espace euclidien E^{n+1} . Dans le dernier chapitre on étudie les champs spéciaux des algèbres de déformation sur le fibre tangent TM .

§ 0. Introduction

Soit M une variété C^∞ -différentiable et à n dimensions. On note par $\mathcal{F}(M)$ -l'anneau des fonctions réelles de classe C^∞ sur M et par $\mathcal{X}(M)$ -le $\mathcal{F}(M)$ -module des champs de vecteurs.

Supposons que la variété M est munie d'un couple de connexions linéaires $(\nabla, \bar{\nabla})$. Si l'on définit le produit de deux champs de vecteurs $X, Y \in \mathcal{X}(M)$ par la formule :

$$(0.1) \quad X * Y = \bar{\nabla}_X Y - \nabla_X Y,$$

alors $\mathcal{X}(M)$ devient une $\mathcal{F}(M)$ -algèbre. Cette algèbre s'appelle l'algèbre de déformation de la paire de connexions $(\nabla, \bar{\nabla})$ et sera notée par $\mathcal{U}(M, \nabla, \bar{\nabla})$ [7].

§ 1. Des champs spéciaux

Considérons deux connexions linéaires $\nabla, \bar{\nabla}$ sur la variété M et soit $A = \bar{\nabla} - \nabla$ le champ tensoriel de déformation.

Définition 1.1. Un champ $X \in \mathcal{X}(M)$ s'appelle champ spécial dans l'algèbre de déformation $\mathcal{U}(M, \nabla, \bar{\nabla})$, si pour tout $Z \in \mathcal{X}(M)$ on a la relation :

$$(1.1) \quad A(Z, X) = 0$$

Proposition 1.2. Soit $X \in \mathcal{X}(M)$. Les propriétés suivantes sont équivalentes :

- (i) X est un champ spécial dans l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$.
- (ii) Si X est un champ ∇ -concourant alors X est champ $\bar{\nabla}$ -concourant.

Démonstration : (i) $\Rightarrow$ (ii) Le champ X est ∇ -concourant [2] si pour tout $Z \in \mathcal{X}(M)$ on a la relation :

$$(1.2) \quad \nabla_Z X = Z$$

De (1.1) et (1.2) il résulte :

$$(1.3) \quad \bar{\nabla}_Z X = Z, \quad (\forall) Z \in \mathcal{X}(M),$$

donc X est champ $\bar{\nabla}$ -concourant.

(ii) $\Rightarrow$ (i) De (1.2) et (1.3) il résulte sans difficulté (1.1).

Proposition 1.3. L'ensemble $S(M)$ des champs spéciaux dans l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$ est un $\mathcal{F}(M)$ -sousmodule du module $\mathcal{X}(M)$.

Démonstration : Soient $X_1, X_2 \in S(M)$, donc pour tout $Z \in \mathcal{X}(M)$ on a :

$$(1.1') \quad A(Z, X_1) = 0, \quad A(Z, X_2) = 0$$

De (1.1') il résulte :

$$(1.1'') \quad A(Z, X_1 + X_2) = 0, \quad (\forall) Z \in \mathcal{X}(M)$$

et

$$(1.1''') \quad A(Z, hX_1) = 0, \quad (\forall) h \in \mathcal{F}(M), \quad (\forall) Z \in \mathcal{X}(M)$$

donc $X_1 + X_2, hX_1 \in S(M)$

Proposition 1.4 Tout champ spécial dans l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$ est un élément nilpotent d'indice 2.

Démonstration : Soit $X \in S(M)$, donc nous avons (1.1) pour tout $Z \in \mathcal{X}(M)$. En particulier, pour $Z = X$ de (1.1) on a :

$$(1.1^{IV}) \quad A(X, X) = 0$$

ce que nous montre que X est un élément nilpotent d'indice 2 dans l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$.

Définition 1.5 Soient $X \in S(M)$ et $p \in M$ tel que $X_p \neq 0$. La direction :

$$d_p = \{rX_p \mid r \in R - \{0\}\}$$

s'appelle direction spéciale.

Soit $X \in S(M)$. Si $X_q \neq 0$, $(\forall) q \in M$, alors X s'appelle champ de directions spéciales dans l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$. Les trajectoires des champs de directions spéciales sont appelées courbes spéciales.

Remarque 1.6 Soit A_{jk}^i , resp. X^i , les composantes de A , resp. X , dans un système de coordonnées locales. Alors la relation (1.1) s'écrit en coordonnées locales :

$$(1.1^V) \quad A_{jk}^i X^k = 0$$

En employant (1.1^v) on obtient le système différentiel d'équations des courbes spéciales de l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$:

$$(1.4) \quad A_{jk}^i \frac{dx^k}{dt} = 0$$

Proposition 1.7 Les propriétés suivantes sont équivalentes :

(i) Tous les éléments de l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$ sont des champs spéciaux.

(ii) Par tout $p \in M$ passe une courbe spéciale tangente à une direction donnée.

(iii) $\bar{\nabla} = \nabla$.

Démonstration : (i) $\Rightarrow$ (iii) De $A(Z, X) = 0$, ($\forall$) $X, Z \in \mathcal{X}(M)$ il résulte $A = 0$, c'est-à-dire $\bar{\nabla} = \nabla$.

(iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) Evidemment.

Exemples 1.8 Soit M une variété différentiable de classe C^∞ . On dit que deux métriques pseudoriemannniennes g et $\bar{g}$ sur M sont conformément équivalentes s'il existe une fonction $u \in \mathcal{F}(M)$ tel que $\bar{g} = e^{2u}g$. Par structure conforme sur M , on comprend une classe $\hat{g}$ de métriques conformément équivalentes. La paire $(M, \hat{g})$ s'appelle une variété conforme.

Sur la variété conforme $(M, \hat{g})$, nous considérons les connexions linéaires $\nabla, \bar{\nabla}$ associées aux $g, \bar{g} \in \hat{g}$. Soit $A = \bar{\nabla} - \nabla$ le champ tensoriel de déformation. Pour tout $X, Y, Z \in \mathcal{X}(M)$ on a la relation [3]:

$$(1.5) \quad X(u)g(Y, Z) - Y(u)g(X, Z) + Z(u)g(X, Y) = g(A(X, Z), Y)$$

Soit X un champ spécial dans l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$, donc nous avons la relation (1.1) pour tout $Z \in \mathcal{X}(M)$. De (1.1) et (1.5) on a :

$$(1.6) \quad X(u)g(Y, Z) - Y(u)g(X, Z) + Z(u)g(X, Y) = 0, \quad (\forall) Y, Z \in \mathcal{X}(M)$$

Si nous notons par g_{ij} , resp. X^i les composantes du g , resp. X , alors, dans les coordonnées locales la relation (1.6) s'écrit :

$$(1.6') \quad X^i u_i g_{jk} - u_j g_{ik} X^i + u_k g_{ij} X^i = 0$$

où $u_i = \frac{\partial u}{\partial x^i}$. En multipliant (1.6') par g^{jk} on obtient :

$$(1.7) \quad X^i u_i = 0$$

De (1.6') et (1.7) on a

$$(1.8) \quad X^i V^j - X^j V^i = 0,$$

où $V^i = g^{ij} u_j$ sont les composantes d'un champ de vecteurs $V \in \mathcal{X}(M)$. De (1.8) il résulte que les champs X et V sont colinéaires.

Supposons que les métriques $g, \bar{g} \in \hat{g}$ diffèrent par un facteur non-constant, c'est-à-dire la fonction u n'est pas une constante. Alors, les trajectoires de champ ayant pour composantes $g^{ij} \frac{\partial u}{\partial x^j}$ sont des courbes spéciales de l'algèbre $\mathcal{U}(M, \nabla, \bar{\nabla})$.

§ 2. Les courbes spéciales d'une hypersurface

Soit M une hypersurface dans l'espace euclidien E^{n+1} . Soit g, b , resp. h , la première, la deuxième, resp. la troisième forme fondamentale de M . Supposons que b est nondégénérée. Soient $\nabla(I), \nabla(II)$ et $\nabla(III)$ les connexions linéaires associées à, g, b et h . Notons :

$$(2.1) \quad \begin{aligned} A &= \nabla(I) - \nabla(II), \quad A' = \nabla(II) - \nabla(III), \\ A'' &= \nabla(I) - \nabla(III) \end{aligned}$$

Lemme 2.1 Pour tout $X, Y, Z \in \mathcal{X}(M)$ on a :

$$(2.2) \quad \begin{aligned} b(A(X, Y), Z) &= b(A'(X, Y), Z) = 2b(A''(X, Y), Z) = \\ &= -\frac{1}{2} (\nabla(I)_X b)(Y, Z) \end{aligned}$$

Démonstration : Voir [1], [3], [5].

Proposition 2.2 Soit M une hypersurface dans l'espace euclidien E^{n+1} . Supposons que la deuxième forme fondamentale est nondégénérée. Les propriétés suivantes sont équivalentes :

- (i) Les $\nabla(I)$ -géodésiques sont $\nabla(II)$ -géodésiques.
- (ii) Les $\nabla(II)$ -géodésiques sont $\nabla(I)$ -géodésiques.
- (iii) $\nabla(I)_X b = 0, \quad (\forall) X \in \mathcal{X}(M)$
- (iv) $\nabla(I) = \nabla(II)$
- (v) Tous les éléments de l'algèbre $\mathcal{U}(M, \nabla(I), \nabla(II))$ sont des champs spéciaux.
- (vi) Tous les éléments de l'algèbre $\mathcal{U}(M, \nabla(I), \nabla(II))$ sont des éléments nilpotents d'indice 2.
- (vii) M est une sphère.

Démonstration : On emploie [5] (Théorème D) et la Proposition 1.7.

Remarque 2.3 En employant le Lemme 2.1 on obtient des résultats analogues pour l'algèbre $\mathcal{U}(M, \nabla(II), \nabla(III))$ et pour l'algèbre $\mathcal{U}(M, \nabla(III), \nabla(I))$.

Remarque 2.4 Les relations (2.2) s'écrivent en coordonnées locales :

$$(2.2) \quad A_{jk}^i = A_{jk}'^i = 2A_{jk}''^i = -\frac{1}{2} b^{is} \nabla(I)_s b_{jk}$$

où $A_{jk}^i, A_{jk}'^i, A_{jk}''^i$ resp. b_{jk} sont les composantes de A, A', A'' resp. b et où $b^{is} b_{sj} = \delta_j^i$.

Employant les relations (2.2') on obtient que les algèbres $\mathcal{U}(M, \nabla(I), \nabla(II))$, $\mathcal{U}(M, \nabla(II), \nabla(III))$ et $\mathcal{U}(M, \nabla(III), \nabla(I))$ ont les mêmes courbes spéciales. Ces courbes s'appellent *des courbes spéciales de l'hypersurface M*. Employant (2.2') nous obtenons le système différentiel d'équations des courbes spéciales de l'hypersurface M :

$$(2.3) \quad b^{is} \nabla(I)_s b_{jk} \frac{dx^k}{dt} = 0$$

En multipliant les équations (2.3) par b_{ir} et en sommant on obtient :

$$(2.4) \quad \nabla(I)_r b_{jk} \frac{dx^k}{dt} = 0$$

Proposition 2.5 Soit M une hypersurface dans l'espace euclidien E^{n+1} . Supposons que la deuxième forme fondamentale b est non dégénérée. Les propriétés suivantes sont équivalentes :

(i) Par tout point $p \in M$ passe une courbe spéciale tangente à une direction donnée.

(ii) $\nabla(I)_X b = 0$, ($\forall$) $X \in \mathcal{X}(M)$

Démonstration : On emploie Proposition 1.7 et Proposition 2.3.

Proposition 2.6 Si une courbe spéciale $x^i = x^i(t)$ de l'hypersurface M est $\nabla(I)$ -géodésique alors on a la suite suivante de relations :

$$\frac{dx^{r_1}}{dt} \nabla(I)_{r_1} b_{jk} = 0$$

$$(2.5) \quad \frac{dx^{r_2}}{dt} \frac{dx^{r_1}}{dt} \nabla(I)_{r_2} \nabla(I)_{r_1} b_{jk} = 0$$

.....

$$\frac{dx^{r_t}}{dt} \frac{dx^{r_{t-1}}}{dt} \dots \frac{dx^{r_2}}{dt} \frac{dx^{r_1}}{dt} \nabla(I)_{r_t} \nabla(I)_{r_{t-1}} \dots \nabla(I)_{r_2} \nabla(I)_{r_1} b_{jk} = 0$$

.....

Démonstration : De (2.4) et de l'équations de Codazzi

$$(2.6) \quad \nabla(I)_r b_{jk} = \nabla(I)_j b_{rk}$$

on a

$$(2.7) \quad \frac{dx^{r_1}}{dt} \nabla(I)_{r_1} b_{jk} = 0$$

En prenant les dérivées covariantes par rapport à $\nabla(I)$ et en tenant compte que

$$(2.8) \quad \nabla(I)_i \frac{dx^i}{dt} = 0$$

on obtient sans difficulté (2.5).

Proposition 2.7 Si une ligne asymptotique $x^i = x^i(t)$ de l'hyper-surface M est une courbe spéciale de M alors elle vérifie :

$$(2.9) \quad b_{ij} \frac{dx^i}{dt} \frac{dx^k}{dt} \nabla(I)_k \frac{dx^j}{dt} = 0$$

Démonstration : On a [6]

$$b_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$$

En prenant les dérivées covariantes par rapport à $\nabla(I)$ et en tenant compte que

$$\nabla(I)_k b_{ij} \frac{dx^k}{dt} = 0,$$

on obtient sans difficulté (2.9).

Remarque 2.8 Supposons que $\nabla(I)_i b_{jk} = \rho_i b_{jk}$ où ρ_i sont les composantes d'une 1-forme sur la hypersurface M . Alors le système (2.4) se réduit à :

$$\rho_i \frac{dx^i}{dt} = 0$$

Exemple 2.9 Soit M une surface dans l'espace euclidien E^3 , donnée par :

$$\begin{aligned} x &= (r' + r \cos x^1) \cos x^2 \\ y &= (r' + r \cos x^1) \sin x^2 \\ z &= r \sin x^1 \end{aligned}$$

où $r', r = \text{const.}$, $r' > r > 0$, $x^1 \in R - \left\{ (2k+1) \frac{\pi}{2} \right\}$, $x^2 \in R$, $k \in Z$

La connexion $\nabla(I)$ a les composantes :

$$\begin{vmatrix} 1 \\ 11 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 \\ 12 \end{vmatrix} = \begin{vmatrix} 1 \\ 21 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 \\ 22 \end{vmatrix} = \frac{1}{r} (r' + r \cos x^1) \sin x^1,$$

$$\begin{vmatrix} 2 \\ 11 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 \\ 12 \end{vmatrix} = \begin{vmatrix} 2 \\ 21 \end{vmatrix} = \frac{-r \sin x^1}{r' + r \cos x^1}, \quad \begin{vmatrix} 2 \\ 22 \end{vmatrix} = 0$$

La connexion $\nabla(III)$ a les composantes :

$$\left| \widetilde{1} \right|_{11} = 0, \quad \left| \widetilde{1} \right|_{12} = \left| \widetilde{1} \right|_{21} = 0, \quad \left| \widetilde{1} \right|_{22} = \sin x^1 \cos x^1,$$

$$\left| \widetilde{2} \right|_{11} = 0, \quad \left| \widetilde{2} \right|_{12} = \left| \widetilde{2} \right|_{21} = -\operatorname{tg} x^1, \quad \left| \widetilde{2} \right|_{22} = 0$$

Le champ tensoriel $A'' = \nabla(I) - \nabla(III)$ a les composantes :

$$A''_{11} = 0, \quad A''_{12} = A''_{21} = 0, \quad A''_{22} = \frac{r'}{r} \sin x^1$$

$$A''_{11}{}^2 = 0, \quad A''_{12}{}^2 = A''_{21}{}^2 = \frac{r' \sin x^1}{(r' + r \cos x^1) \cos x^1}, \quad A''_{22}{}^2 = 0$$

Le système différentiel d'équations des courbes spéciales de la surface M :

$$\begin{cases} A''_{12} \frac{dx^1}{dt} = 0 \\ A''_{12} \frac{dx^2}{dt} = 0 \\ A''_{22} \frac{dx^2}{dt} = 0 \end{cases}$$

nous montre que :

a) Dans les points des cercles équateuriels $x^1 = k\pi$, $k \in \mathbb{Z}$, les directions spéciales sont indéterminés.

b) Les cercles méridiennes (sauf les points paraboliques de la surface M) constituent la famille des courbes spéciales de la surface M .

§ 3. Des champs spéciaux sur TM

Soit M une variété différentielle et $T_x M$ l'espace tangent dans un point $x \in M$. Un système de coordonnées (U, x^h) qui couvre M introduit sur le fibre tangent $TM = \bigcup_{x \in M} T_x M$ un système de coordonnées

$\{\pi^{-1}(U), (x^h, y^h)\}$ où $\pi: TM \rightarrow M$ est la projection naturelle. En ce qui suit on utilisera les coordonnées $\{x^h, y^h\}$ sur TM .

Soit ∇ une connexion sur M , ayant pour composantes locales Γ_{jk}^i .

Un champ de vecteurs $X \in \mathcal{X}(M)$ ayant pour composantes locales X^h induit sur TM trois champs de vecteurs :

1) le lift vertical $X^v = \begin{pmatrix} 0 \\ X^i \end{pmatrix}$

2) le lift complet $X^c = \begin{pmatrix} X^i \\ y^j \frac{\partial X^h}{\partial x^j} \end{pmatrix}$

3) le lift horizontal $X^h = \begin{pmatrix} X^h \\ -y^h \Gamma_{hj}^i X^j \end{pmatrix}$

La connexion ∇ induit sur TM deux connexions [4] :

(1) le lift complet

$$\nabla_X^c Y^c = (\nabla_X Y)^c, \quad (\forall) X, Y \in \mathcal{X}(M)$$

ayant les composantes locales :

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h, \quad \bar{\Gamma}_{i\bar{j}}^h = 0, \quad \bar{\Gamma}_{i\bar{j}}^h = 0, \quad \bar{\Gamma}_{i\bar{j}}^h = 0,$$

$$\bar{\Gamma}_{ij}^{\bar{h}} = y^k \frac{\partial \Gamma_{ij}^h}{\partial x^k}, \quad \bar{\Gamma}_{i\bar{j}}^{\bar{h}} = \Gamma_{ij}^h, \quad \bar{\Gamma}_{i\bar{j}}^{\bar{h}} = \Gamma_{ij}^h, \quad \bar{\Gamma}_{i\bar{j}}^{\bar{h}} = 0$$

où $\bar{i}, \bar{j}, \bar{k}, \dots = \bar{1}, \bar{2}, \dots, \bar{n}$; $\bar{i} = i + n$.

Nous avons :

$$\nabla_X^c V^{Y^v} = 0, \quad \nabla_X^c V^{Y^h} = 0, \quad \nabla_X^c H^{Y^v} = (\nabla_X Y)^v$$

$$\nabla_X^c H^{Y^h} = (\nabla_X Y)^h + \gamma(R(\quad, X)Y), \quad \nabla_X^c C^{Y^v} = (\nabla_X Y)^v,$$

$$\nabla_X^c V^{Y^c} = (\nabla_X Y)^v, \quad (\forall) X, Y \in \mathcal{X}(M)$$

où R est le tenseur de courbure de ∇ et

$$\gamma(R(\quad, X)Y) = y^k R_{ijk}^h X^i Y^j \frac{\partial}{\partial x^k}$$

(2) le lift horizontal :

$$\nabla_X^h H^{Y^h} = (\nabla_X Y)^h, \quad \nabla_X^h H^{Y^v} = (\nabla_X Y)^v, \quad \nabla_X^h V^{Y^h} = 0,$$

$$\nabla_X^h V^{Y^v} = 0, \quad (\forall) X, Y \in \mathcal{X}(M)$$

ayant les composantes locales :

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h, \quad \bar{\Gamma}_{i\bar{j}}^h = 0, \quad \bar{\Gamma}_{i\bar{j}}^h = 0, \quad \bar{\Gamma}_{i\bar{j}}^h = 0$$

$$\bar{\Gamma}_{ij}^h = y^k \frac{\partial \Gamma_{ij}^h}{\partial x^k} - y^k R_{ijk}^h, \quad \bar{\Gamma}_{ij}^h = \Gamma_{ij}^h, \quad \bar{\Gamma}_{ij}^{\bar{h}} = \Gamma_{ij}^{\bar{h}},$$

$$\bar{\Gamma}_{ij}^h = 0$$

Nous avons :

$$\nabla_X^H C^{Y^V} = (\nabla_X Y)^V, \quad \nabla_X^H V^{Y^V} = (\nabla_X Y)^V,$$

$$\nabla_X^H C^{Y^C} = (\nabla_X Y)^C - \gamma(R(\quad, X)Y), \quad (\forall) X, Y \in \mathcal{X}(M).$$

I) Soit $A_1 = \nabla^C - \nabla^H$. On a [4] :

$$A_1(X^V, Y^V) = 0, \quad A_1(X^V, Y^H) = 0, \quad A_1(X^H, Y^V) = 0,$$

$$A_1(X^V, Y^C) = 0, \quad A_1(X^C, Y^V) = 0,$$

$$A_1(X^C, Y^C) = \gamma(R(\quad, X)Y) = A_1(X^H, Y^H).$$

On obtient :

Proposition : 3.1 Soit $X \in \mathcal{X}(M)$. Alors $X^V \in \mathcal{U}(TM, \nabla^C, \nabla^H)$ est de champ spécial.

II) Supposons dans la suite que la variété M soit munie d'une paire de connexions linéaires $(\nabla, \bar{\nabla})$. Si l'on note $A = \bar{\nabla} - \nabla$, $A_2 = \bar{\nabla}^C - \nabla^C$, alors nous avons [4] :

$$A_2(X^V, Y^V) = 0, \quad A_2(X^V, Y^C) = (A(X, Y))^V,$$

$$A_2(X^C, Y^V) = (A(X, Y))^V$$

$$A_2(X^C, Y^C) = (A(X, Y))^C$$

De même, on a [4] :

$$A_2(X^H, Y^H) = (A(X, Y))^H + (\nabla_Y A)(X^H, Y^H),$$

où le lift horizontal est considéré par rapport à ∇ et

$$\nabla_Y A = y^k \nabla_k A_{ij}^h \frac{\partial}{\partial y^h} \otimes dx^i \otimes dx^j$$

(où A_{ij}^h sont les composantes locales de A). Il résulte :

Proposition 3.2 Pour tout $X \in \mathcal{S}(M)$ il résulte que :

- i) $X^V \in \mathcal{U}(TM, \nabla^C, \bar{\nabla}^C)$ est champ spécial.
- ii) $X^C \in \mathcal{U}(TM, \nabla^C, \bar{\nabla}^C)$ est un champ spécial.

iii) Si $(\nabla_\gamma A)(Z^H, X^H) = 0$ pour tout $Z \in \mathcal{X}(M)$, alors $X^H \in \mathcal{U}(TM, \nabla^C, \bar{\nabla}^C)$ est un champ spécial.

III) Si nous notons $A_3 = \bar{\nabla}^C - \nabla^H$ alors nous avons les formules [4]:

$$A_3(X^V, Y^V) = 0, \quad A_3(X^V, Y^C) = (A(X, Y))^V,$$

$$A_3(X^C, Y^V) = (A(X, Y))^V,$$

$$A_3(X^C, Y^C) = (A(X, Y))^C + \gamma(R(\quad, X)Y)$$

On a :

Proposition 3.3 Soit $X \in S(M)$. Alors :

- i) $X^V \in \mathcal{U}(TM, \nabla^H, \bar{\nabla}^C)$ est un champ spécial.
- ii) Si $\gamma(R(\quad, X)X) = 0$, alors $X^C \in \mathcal{U}(TM, \nabla^H, \bar{\nabla}^C)$ est un champ spécial.

IV) Si nous notons $A_4 = \bar{\nabla}^H - \nabla^H$, alors nous avons [4]:

$$A_4(X^V, Y^V) = 0, \quad A_4(X^V, Y^C) = (A(X, Y))^V = A_4(X^C, Y^V)$$

$$A_4(X^C, Y^C) = (A(X, Y))^C + \gamma(R(\quad, X)Y) - \gamma(\bar{R}(\quad, X)Y)$$

ou $\bar{R}$ est le tenseur de courbure de $\bar{\nabla}$. Il en résulte :

Proposition 3.4 Soit $X \in S(M)$. Alors :

- i) $X^V \in \mathcal{U}(TM, \nabla^H, \bar{\nabla}^H)$ est un champ spécial.
- ii) Si $\gamma(R(\quad, Z)X) = \gamma(\bar{R}(\quad, Z)X)$, $(\forall Z \in \mathcal{X}(M))$, alors $X^C \in \mathcal{U}(TM, \nabla^H, \bar{\nabla}^H)$ est un champ spécial.

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A NOTE ON N-CONTEXTUAL GRAMMARS

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In this paper one studies the families of languages generated by n -contextual grammars, generalized n -contextual grammars, inner n -contextual grammars, n -contextual grammars with choice and, respectively, context-sensitive adjunct grammars on the alphabet of one letter. The interrelations to the Chomsky hierarchy (on the one-letter alphabet) and the length set of languages are investigated.

1. Preliminaries

We assume that the reader to be familiar with Salomaa's monograph [7]. We shall specify here some necessary notions, and notations.

We denote by V^* the free monoid generated by the alphabet V with the unity λ . The length of a string x is denoted by $|x|$. The length set of a language is defined as the set of integers n such that $n = |x|$ for some x in L . $\mathcal{L}_i, i = 0, 1, 2, 3$, rec denote the families of recursively enumerable, context-sensitive, context-free, regular and, respectively, recursive languages.

We denote by $[A]^n$ the cartesian product of n exemplars of A , by 2^A the set of all subsets of A and by $\Rightarrow^*$ the reflexive and transitive closure of a relation $\Rightarrow$.

Definition 1 [2] *An n -contextual grammar (briefly, an n -CG) ($n > 2$) is a quadruple*

$$G = \langle V, B, C, f \rangle$$

where $B \subset V^, C \subset [V^*]^{n-1}$ are finite and $f: [V^*]^n \rightarrow 2^C$ is effectively computable.*

The language generated by G is

$$L(G) = \{x \in V^* \mid b \Rightarrow^* x \text{ for some } b \text{ in } B\}$$

where for x, y in V^* , $x \Rightarrow y$ iff there exists $x_1, x_2, \dots, x_n$ in V^* , $(u_1, \dots, u_{n-1})$ in $f(x_1, \dots, x_n)$ such that $x = x_1 \dots x_n$ and $y = x_1 u_1 \dots x_{n-1} u_{n-1} x_n$.

A language L is called *n -contextual language* (briefly *n -CL*) iff there is an n -CG generating L . We denote by $\mathcal{F}(n)$ the family of n -CL's.

Definition 2 [3] *An n -contextual grammar with choice (briefly, an n -CCG) ($n \geq 3$) is an n -CG $\langle V, B, C, f \rangle$ where the condition $f(x_1, \dots, x_n) \neq \emptyset$ implies $x_1 = x_n = \lambda$.*

A language generated by an n -CCG is called n -contextual language with choice (briefly, n -CCL). We denote by $\mathcal{C}(n)$ the family of all n -CCL's.

Definition 3 [3] *An inner n -contextual grammar (briefly, an n -ICG) ($n \geq 3$) is an n -CG $\langle V, B, C, f \rangle$ where for arbitrary $x_1, \dots, x_n$ in V^* , $f(x_1, x_2, \dots, x_{n-1}, x_n) = f(\lambda, x_2, \dots, x_{n-1}, \lambda)$*

A language generated by an n -ICG is termed inner n -contextual language (briefly, n -ICL). We denote by $\mathcal{I}(n)$ the family of all n -ICL's.

Definition 4 [3] *A generalized n -contextual grammar (briefly, an n -GCG) ($n \geq 3$) is an $(2n + 1)$ -tuple*

$$G = \langle V, B_1, \dots, B_{n-2}, D_1, \dots, D_{n-2}, C \rangle.$$

where $B_1, \dots, B_{n-2}, D_1, \dots, D_{n-2}$ are finite subset of V^* and $C \subset [V^*]^{n-1}$ is finite.

The language generated by G is the set

$$L(G) = B_1 \dots B_{n-2} \cup \{u_{11} \dots u_{1m} b_1 y_1^m u_{2m} \dots u_{21} b_2 y_2^m \dots$$

$$\dots b_{n-2} y_{n-2}^m u_{n-1m} \dots u_{n-11} \mid m \geq 1, b_i \in B_i$$

$$y_i \in D_i, 1 \leq i \leq n-2, (u_{1j}, \dots, u_{n-1j}) \in C, 1 \leq j \leq m\}$$

and is called generalized n -contextual language (briefly, n -GCL). We denote by $\mathcal{G}(n)$ the family of all n -GCL's.

Proposition 1 ([2], [3]) *The following proper inclusions hold true for any $n \geq 2$*

- (i) $\mathcal{F}(n) \cup \mathcal{C}(n+1) \cup \mathcal{I}(n+1) \cup \mathcal{G}(n+1) \subset \mathcal{F}(n+1) \subset \mathcal{L}_{\text{rec}}$
- (ii) $\mathcal{C}(n+1) \subset \mathcal{C}(n+2)$
- (iii) $\mathcal{I}(n+1) \subset \mathcal{I}(n+2)$
- (iv) $\mathcal{G}(n+1) \subset \mathcal{G}(n+2)$
- (v) $\mathcal{F}(n) \subset \mathcal{C}(n+2)$.

2. Results

Proposition 2. *On the one-letter alphabet, for any $n \geq 2$, any n -CL is 2 -CL.*

Proof.: Let G be an n -CG over $\{a\}$

$$G = \langle \{a\}, B, C, f \rangle$$

Then $L(G)$ is generated by the 2 -CG

$$G_1 = \langle \{a\}, B, C_1, f_1 \rangle$$

where

$$C_1 = \{x \mid x = x_1 \dots x_{n-1} \text{ for some } (x_1, \dots, x_{n-1}) \text{ in } C\}$$

$$f_1(x, y) = \cup \{f(x_1, \dots, x_n) \mid x_1 \dots x_n = xy\}$$

Proposition 3. On the one-letter alphabet, for any $n \geq 3$, any $n - ICL$ is $3 - ICL$.

Proof. Let G be an $n - ICG$ over $\{a\}$

$$G = \langle \{a\}, B, C, f \rangle$$

Then $L(G)$ is also generated by $3 - ICG$

$$G_1 = \langle \{a\}, B, C_1, f_1 \rangle$$

where

$$C_1 = \{(x, \lambda) \mid x = u_1 \dots u_{n-1} \text{ for some } (u_1, \dots, u_{n-1}) \text{ in } C\}$$

$$f_1(x, y, z) = \cup \{f(x, y_1, \dots, y_{n-2}, z) \mid y = y_1 \dots y_{n-2}\}$$

Proposition 4. On the one-letter alphabet, for $n \geq 3$, any $n - CCL$ is $3 - CCL$.

Proof. For a $n - CCG$

$$G = \langle \{a\}, B, C, f \rangle$$

we can construct an equivalent $3 - CCG$ as follows

$$G_1 = \langle \{a\}, B, C_1, f_1 \rangle$$

where

$$C_1 = \{(u, \lambda) \mid u = u_1 \dots u_{n-1} \text{ for some } (u_1, \dots, u_{n-1}) \text{ in } C\}$$

$$f(\lambda, x, \lambda) = \cup \{f(\lambda, x_1, \dots, x_{n-2}, \lambda) \mid x = x_1 \dots x_{n-2}\}$$

Proposition 5. On the one-letter alphabet, for any $n \geq 3$, any $n - GCL$ is $3 - GCL$.

Proof. Let $G = \langle \{a\}, B_1, \dots, B_{n-2}, D_1, \dots, D_{n-2}, C \rangle$ be an $n - GCG$. We can verify that the following $3 - GCG$

$$G_1 = \langle \{a\}, B, D, C_1 \rangle$$

where $B = B_1 \dots B_{n-2}$, $D = D_1 \dots D_{n-2}$ and $C_1 = \{(u, \lambda) \mid u = u_1 \dots u_{n-1} \text{ for some } (u_1, \dots, u_{n-1}) \text{ in } C\}$ generates $L(G)$.

Proposition 6. [6] On the alphabet of one letter, a language is regular iff it is generated by a $3 - ICG$. If L is regular, there is a $3 - ICG$ generating it such that there is an integer h , $f(x, y, z) \neq \emptyset$ implies $|y| \leq h$.

Proposition 7. On the one-letter alphabet, any $3 - GCL$ is regular. There exist regular languages (on one letter) which are not $3 - GCL$.

Proof. Let L be generated by the $3 - GCG$

$$G = \langle \{a\}, B, D, C \rangle$$

then L is generated by the type 3 grammar with the following productions. where S is the initial symbol

$$S \rightarrow b, \quad b \in B$$

$$S \rightarrow bX, \quad b \in B$$

$$\begin{cases} X \rightarrow uvY_i, \\ Y_i \rightarrow d_i, \\ Y_i \rightarrow uvd_iY_i, \end{cases} \quad \begin{matrix} D = \{d_1, \dots, d_m\} \\ (u, v) \in C, 1 \leq i \leq m. \end{matrix}$$

therefore every 3 - GCL is regular.

On the other hand, the regular language

$$L = \{a\} \cup \{a^{5^n} \mid n \geq 1\}$$

cannot be generated by any 3 - GCG.

We assume that L is generated by a 3 - GCG

$$G = \langle \{a\}, B, D, C \rangle$$

Since a is in L and it is the smallest length string, a must be in B . Because of L is infinite, there exist d in D and (u, v) in C such that $|d| > 0$. Then for any $m \geq 0$, $ad^m u^m v^m$ is in $L(G)$. Easily to see that there are many m 's such that $ad^m u^m v^m$ do not belong to L . Contradiction.

Proposition 8. On the one-letter alphabet, the following assertions hold true :

- (i) A language is n - CL for some n iff it is m - CCL for some m .
- (ii) Any regular language is 2 - CL and there exist 2 - CL's which are not regular.
- (iii) There exist context-sensitive languages which are not n - CL for any $n \geq 2$.

Proof. : (i) is deduced from Proposition 1, 2 and 4.

The first part of (ii) is a corollary of propositions 2, 1 and 6. For the second part, the language

$$L = \{a^n \mid n \neq 2^p, p \geq 1\}$$

is not regular however it is generated by the 2 - CG where

$$f(a^n, \lambda) = \{a\} \text{ iff } n \neq 2^p - 1, p \geq 1,$$

$$f(a^n, \lambda) = \{a^2\} \text{ iff } n = 2^p, p \geq 1,$$

$$f(x, y) = \emptyset \text{ otherwise.}$$

For (iii), we note that the context-sensitive language $L = \{a^{2^n} \mid n \geq 1\}$ is not n - CL for any $n \geq 2$ [2].

Remark 1. On the alphabet with at least two letters, there exist 2 - CL's, 3 - CCL's and 3 - ICL's which are not context-sensitive. However we do not know whether or not on one letter there exist 2 - CL's which are not context-sensitive. It would seem that the answer is affirmative.

If we restrict the consideration on a subclass of 2 - CG's which will be defined later, the corresponding family of languages will be exactly the same of regular languages (on one letter).

Definition 5. ([1], [5]). A context-sensitive adjunct grammar (briefly, a CAG) is defined exactly as a 2 - CG but the derivation is defined as

follows: $x \Rightarrow y$ iff there exist x_1, x_2, x_3, x_4 in V^* , u in $f(x_2, x_3)$ such that $x = x_1x_2x_3x_4$ and $y = x_1x_2ux_3x_4$.

Easily to verify that any language generated by a CAG is also generated by a 2 - CG where

$$f_1(x, y) = \cup \{f(x_2, y_2) \mid x = x_1x_2, y = y_2y_1\}$$

Proposition 9. ([1], [5]) *On the one-letter alphabet, a language is regular iff it is generated by a CAG. Every regular language can be generated by a CAG with the following property: there is an integer $h \geq 0$ such that the condition $f(x, y) \neq \emptyset$ implies $|x| \leq h$ and $y = \lambda$. On the alphabets with at least two letters, there exist non-context-sensitive languages generated by CAG's.*

As regards the length sets of languages, it turns out that the situation is not similiary as in the case of one-letter.

Proposition 10. *For a set of integers L , the following strict implications hold true*

$$\begin{array}{c} (4) \\ \uparrow \\ (1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8) \Leftrightarrow (9) \\ \downarrow \\ (5) \end{array}$$

where

- (1) L is the length set of a 3 - GCL.
- (2) L is the length set of an n - GCL.
- (3) L is the length set of a regular language.
- (4) L is the length set of an n - ICL.
- (5) L is the length set of a language generated by a CAG.
- (6) L is the length set of a 2 - CL.
- (7) L is the length set of an n - CL.
- (8) L is the length set of a 3 - CCL.
- (9) L is the length set of an n - CCL.

Proof. The equivalences $(1) \Leftrightarrow (2)$ and $(6) \Leftrightarrow (7) \Leftrightarrow (8) \Leftrightarrow (9)$ are deduced by Propositions 1, 2, 4, 5, 8 and by the fact that the homomorphic image of an n - GCL, an n - CL, an n - CCL is n - GCL, n - CL, respectively, n - CCL ([2], [4]).

The implication $(2) \Rightarrow (3)$ is implied by proposition 7.

Remark that the set $L = \{n \mid n \neq 2^p, p \geq 1\}$ is not the length set of any regular language.

L is the length set of the language

$$L_1 = \{a^n \mid n \neq 2^p, p \geq 1\}$$

which is 2 - CL (see the proof of proposition 8). Therefore the strict implication $(3) \Rightarrow (6)$ holds true.

L is also the length set of the language

$$L_2 = \{ab^{n-2}a \mid n > 2, n \neq 2^p, p \geq 1\}$$

which is generated by the CAG with $B = \{aba\}$ and

$$f(ab^n, a) = \{b\} \text{ iff } n \neq 2^p - 3, p \geq 1,$$

$$f(ab^n, a) = \{b^2\} \text{ iff } n = 2^p - 3, p \geq 1,$$

$$f(x, y) = \emptyset \text{ otherwise.}$$

Hence we have the strict implication $(3) \Rightarrow (5)$.

L_2 is also generated by the 4-ICG where $B = \{aba\}$ and

$$f(\lambda, ab^n, a, \lambda) = \{b\} \text{ iff } n \neq 2^p - 3, p \geq 1,$$

$$f(\lambda, ab^n, a, \lambda) = \{b^2\} \text{ iff } n = 2^p - 3, p \geq 1,$$

$$f(\lambda, x, y, \lambda) = \emptyset \text{ otherwise}$$

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ON LETTER-RECOGNITION IN GREEK ALPHABET

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The aim of this paper is to establish a strategy for a rapid recognition of a capital letter in the Greek alphabet. This strategy is given in a graph. For establishing this graph we shall use the method described in the comment 5.3 of the book [1].

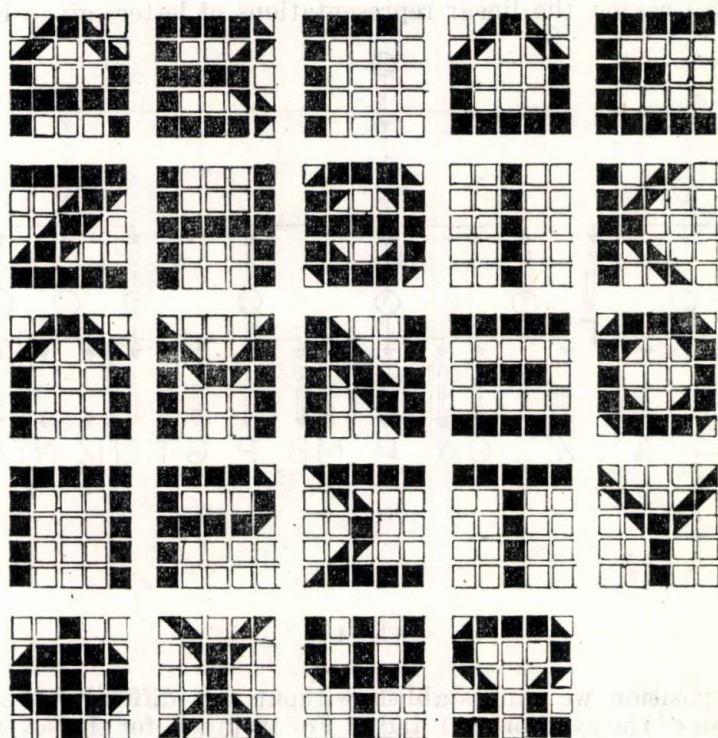


Fig. 1

Let us start by quantifying the capital letters of the Greek alphabet. For doing that we shall use, for each letter 25 squares of three types :
 — white squares

- black squares
- semi-black squares

The quantification of letters is given in Fig. 1. Now we want to know what positions (from the total of 25 positions) must be checked, and in what order for recognizing the type of letter belonging to the Greek

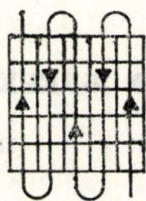


Fig. 2

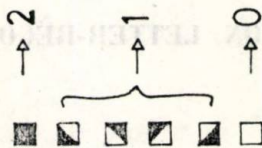


Fig. 3

alphabet. We shall use the entropic algorithm for recognition according to which, at each step we shall prefer, to check a position removing the largest amount of uncertainty or, equivalently, supplying the largest amount of information. Following the pattern given in Fig. 2 and using the symbols 0, 1 and 2 respectively for the three kinds of small squares (see Fig. 3.) we get the linear representations of letters given in Fig. 4.

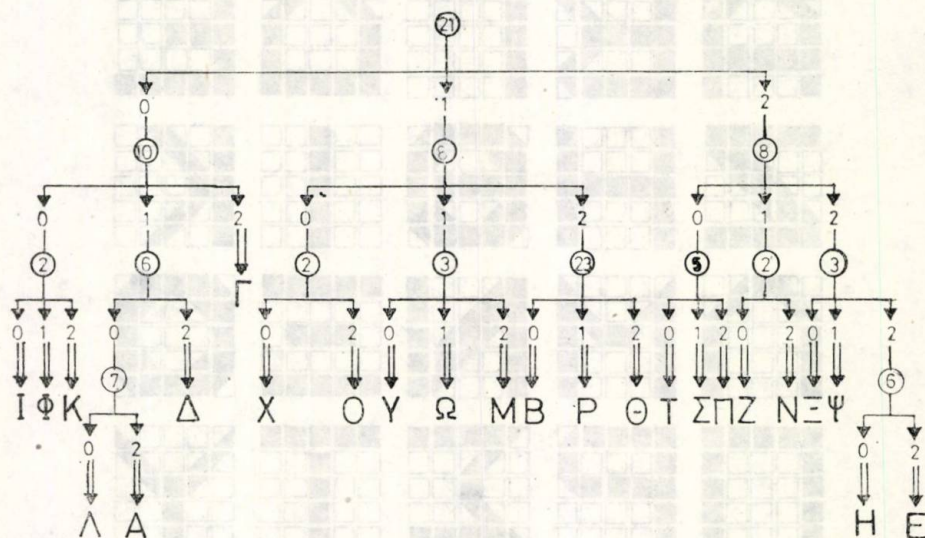


Fig. 4

For each position we can establish without any difficulty the random distribution of the symbols 0, 1 and 2. For instance, for the seventh position we obtain corresponding random distribution:

$$\begin{pmatrix} 0 & 1 & 2 \\ 15 & 5 & 4 \end{pmatrix}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
A	→	0	1	2	2	2	0	2	0	1	1	2	0	0	2	0	0	2	0	1	1	0	1	2	2	2
B	→	2	2	2	2	2	0	2	0	2	2	0	2	0	2	2	1	2	1	2	1	1	0	1	1	
C	→	2	2	2	2	2	0	0	0	2	2	0	0	0	0	0	0	0	0	2	0	0	0	0	0	
D	→	0	1	2	2	2	2	0	0	1	1	2	0	0	0	2	2	0	0	1	1	0	1	2	2	2
E	→	2	2	2	2	2	0	2	0	2	2	0	2	0	2	2	0	0	0	2	2	0	0	0	2	
Z	→	2	0	0	1	2	2	2	1	0	2	2	1	2	1	2	2	0	1	2	2	2	1	0	0	2
H	→	2	2	2	2	2	0	0	2	0	0	0	0	2	0	0	0	0	2	0	0	2	2	2	2	2
Θ	→	1	2	2	2	1	2	1	2	1	2	2	0	2	0	2	2	1	2	1	2	1	2	2	2	1
I	→	0	0	0	0	0	0	0	0	0	2	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0
K	→	2	2	2	2	2	0	1	2	1	0	1	1	0	1	1	1	0	0	0	1	0	0	0	0	0
Λ	→	0	1	2	2	2	0	0	0	1	1	2	0	0	0	0	0	0	0	1	1	0	1	2	2	2
M	→	1	2	2	2	2	0	0	1	1	0	0	0	2	0	0	0	0	1	1	0	1	2	2	2	2
N	→	2	2	2	2	2	0	0	1	2	1	0	1	2	1	0	1	2	1	0	0	2	2	2	2	2
Ξ	→	2	0	0	0	2	2	0	2	0	2	2	0	2	0	2	2	0	2	0	2	2	0	0	0	2
O	→	1	2	2	2	1	2	1	0	1	2	2	0	0	0	2	2	1	0	1	2	1	2	2	2	1
Π	→	2	2	2	2	2	0	0	0	0	2	2	0	0	0	0	0	0	0	0	2	2	2	2	2	2
P	→	2	2	2	2	2	0	0	2	0	2	2	0	2	0	0	0	0	2	0	2	1	2	1	0	0
Σ	→	1	0	0	0	1	2	1	0	1	2	2	1	2	1	2	2	0	0	0	2	2	0	0	0	2
T	→	2	0	0	0	0	0	0	0	0	2	2	2	2	2	2	0	0	0	0	2	2	0	0	0	0
Υ	→	1	1	0	0	0	0	0	1	1	0	0	0	2	2	2	0	0	1	1	0	1	1	0	0	0
Φ	→	0	1	2	1	0	0	2	0	2	0	2	2	2	2	2	0	2	0	2	0	0	1	2	1	0
X	→	1	0	0	0	1	1	1	0	1	1	0	2	2	2	0	1	1	0	1	1	1	0	0	0	1
Ψ	→	2	2	1	0	0	0	0	2	0	0	2	2	2	2	2	0	0	2	0	0	2	2	1	0	0
Ω	→	1	2	1	0	2	2	2	1	0	2	2	0	0	0	0	2	2	1	0	2	1	2	1	0	2

Fig. 5

which means that the verification of the position will remove an amount of uncertainty measured by Shannon entropy :

$$H_7 = \frac{15}{24} \log_2 \frac{15}{24} - \frac{5}{24} \log_2 \frac{5}{24} - \frac{4}{24} \log_2 \frac{4}{24} = 0.919174828$$

Computing the amounts of uncertainty removed by the verification of each position and picking up the position characterized by the maximum value of the corresponding entropy we obtain the final graph given in Fig. 5. If at some step there are two or more than two optimum positions we shall prefer the first one from the left to the right. From Fig. 5 we can see that at most four positions have to be verified for recognizing the corresponding letter. The mean value of the number of verification is 3.125.

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THE CONVERGENCE OF WEIGHTING PROCESS

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SUMMARY

A necessary and sufficient condition for the convergence of weighting process is given.

1. Introduction and Preliminary Definitions

The concept of weighting process was introduced by professor S. Guiaşu in 1975. Conditions which assure the convergence of generalized and iterated weighting processes and sufficient conditions for the converse H-theorem to be true for weighting processes have been also given (see for instance S. Guiaşu (1975)). Also, in 1978, S. Guiaşu proved that the strong converse H-Theorem is true for a certain class of weighting processes.

Let $\mathcal{H} = \{h_1, \dots, h_k\}$ be a finite set. An element of $\mathcal{H}$ can be viewed either as an arbitrary hypothesis or as a state of a given system. Let $\{p_0(h) / h \in \mathcal{H}\}$ be an initial random distribution defined on the set $\mathcal{H}$, satisfying the following conditions:

$$p_0(h) > 0, (h \in \mathcal{H}), \sum_{h \in \mathcal{H}} p_0(h) = 1.$$

Let $\mathcal{U} = \{u_t(h) / h \in \mathcal{H}, t \in I\}$ be a family of non-negative real numbers' $u_t(h) \geq 0$, where the time-interval I may be either the set R^+ of positive real numbers (continuous time), or the set N^* of positive integer numbers (discrete time). We suppose that the family $\mathcal{H}$ is nondegenerate, i.e., for any $t \in I$ there exists $h \in \mathcal{H}$ such that $u_t(h) > 0$.

Definition 1. The weighting process in the general sense describes a stochastic evolution on the $\mathcal{H}$ for which the random distribution of the elements $h \in \mathcal{H}$, at any moment $t \in I$ is given by the expression:

$$p_t(h) = p_0(h) \cdot u_t(h) \cdot \left[\sum_{h \in \mathcal{H}} p_0(h) \cdot u_t(h) \right]^{-1}, \quad (h \in \mathcal{H})$$

Here, $\mathcal{U}$ denotes the family of weights corresponding to the time interval $[0, t]$.

Definition 2. The weighting process is convergent if for any $h \in \mathcal{H}$ there exists the asymptotical probability

$$p_{\infty}(h) = \lim_{t \rightarrow \infty} p_t(h)$$

Let $\tilde{\mathcal{U}} = \{ \tilde{u}_t(h) / h \in \mathcal{H}, t \in N^* \}$ a family of non-negative real numbers.

Definition 3. The sequential weighting process is a random evolution on the set $\mathcal{H}$ such that the random distribution of the elements $h \in \mathcal{H}$ at any moment $t \in N^$ is given by the expression*

$$p_t(h) = p_{t-1}(h) \cdot \tilde{u}_t(h) \cdot \left[\sum_{h \in \mathcal{H}} p_{t-1}(h) \cdot \tilde{u}_t(h) \right]^{-1}, \quad (h \in \mathcal{H})$$

Here, the real numbers $\{ \tilde{u}_t(h) / h \in \mathcal{H} \}$ represent the weights at the moment t .

Definition 4. The iterated weighting process is the weighting process given by

$$p_t(h) = p_0(h) \cdot (u(h))^t \cdot \left[\sum_{h \in \mathcal{H}} p_0(h) \cdot (u(h))^t \right]^{-1}, \quad (h \in \mathcal{H})$$

where the function $u(h)$ is called elementary weight.

S. Guiaşu proved in 1975 that any sequential weighting process can be viewed as a generalized weighting process. In the following, we are going to give a necessary and sufficient condition for the convergence of weighting processes. Some sufficient conditions for the convergence of different types of weighting processes will be given also.

2. The Convergence of the Weighting Process

Let us introduce the following sets :

$$\mathcal{H}_1 = \{ h \in \mathcal{H} / (\exists) t \in I : u_t(h) = 0 \}$$

$$\mathcal{H}'_1 = \{ h \in \mathcal{H} / (\exists) t \in I : p_t(h) = 0 \}$$

$$\mathcal{H}_2 = \{ h_1 \in \mathcal{H} \setminus \mathcal{H}_1 / (\forall) h \notin \mathcal{H}_1, h \neq h_1, \lim_{t \rightarrow \infty} u_t(h) / u_t(h_1) = 0 \}$$

$$\mathcal{H}'_2 = \{ h_1 \in \mathcal{H} \setminus \mathcal{H}'_1 / \lim_{t \rightarrow \infty} p_t(h_1) = 1 \}$$

Then, we get :

Lemma 1 :

a) $\mathcal{H}_1 = \mathcal{H}'_1$

b) $\mathcal{H}_2 = \mathcal{H}'_2$

Proof. Obviously (a) is true. In order to prove (b) let us consider two cases. If $\mathcal{H}_2 = \emptyset$ then $\mathcal{H}'_2 = \emptyset$ because in the contrary there exists $h_1 \notin \mathcal{H}_1$ such that: $\lim_{t \rightarrow \infty} p_t(h_1) = 1$, i.e.,

$$(1) \quad \lim_{t \rightarrow \infty} \sum_{\substack{\bar{h} \notin \mathcal{H}_1 \\ \bar{h} \neq h_1}} p_0(\bar{h}) \cdot u_t(\bar{h}) / u_t(h_1) = 0$$

Since $p_0(\bar{h}) > 0$ and $u_t(\bar{h})/u_t(h_1) > 0$, for every $\bar{h}$, $h_1 \notin \mathcal{H}_1$, by using (1) we obtain: $\lim_{t \rightarrow \infty} u_t(\bar{h})/u_t(h_1) = 0$ for every $\bar{h} \notin \mathcal{H}_1$, $\bar{h} \neq h_1$, i.e., $\mathcal{H}_2 \neq \emptyset$;

which is obviously a contradiction. Hence, if $\mathcal{H}_2 = \emptyset$ then $\mathcal{H}'_2 = \emptyset$. By proceeding in a similar manner, from $\mathcal{H}'_2 = \emptyset$, we obtain $\mathcal{H}_2 = \emptyset$. Therefore, $\mathcal{H}_2 = \emptyset$ if and only if $\mathcal{H}'_2 = \emptyset$.

Now, we suppose $\mathcal{H}_2 \neq \emptyset$. Then, for $h_1 \in \mathcal{H}_2$, there is a positive number t_0 such that for any $t > t_0$, we have

$$(2) \quad p_t(h_1) = p_0(h_1) \cdot \left[\sum_{\substack{\bar{h} \notin \mathcal{H}_1 \\ \bar{h} \neq h_1}} p_0(\bar{h}) \cdot u_t(\bar{h})/u_t(h_1) + p_0(h_1) \right]^{-1}.$$

Since $\lim_{t \rightarrow \infty} u_t(\bar{h})/u_t(h_1) = 0$, by using (2) we get: $\lim_{t \rightarrow \infty} p_t(h_1) = 1$. Hence, $h_1 \in \mathcal{H}'_2$ and $\mathcal{H}_2 \subseteq \mathcal{H}'_2$. Conversely, if $\mathcal{H}'_2 \neq \emptyset$ then one easily can prove that $\mathcal{H}'_2 \subseteq \mathcal{H}_2$. Therefore $\mathcal{H}_2 = \mathcal{H}'_2$.

q.e.d.

Corollary 1. If $\mathcal{H}'_2 \neq \emptyset$ then $\text{card } \mathcal{H}'_2 = 1$.

Proof: If there are $h_1, h_2 \in \mathcal{H}'_2$, such that $h_1 \neq h_2$, then

$$(3) \quad \lim_{t \rightarrow \infty} p_t(h_1) = 1, \quad \lim_{t \rightarrow \infty} p_t(h_2) = 1$$

But, for every $t \in I$ we have

$$(4) \quad \sum_{\substack{h \in \mathcal{H} \\ h \notin \mathcal{H}_1}} p_t(h) = 1.$$

As it can be easily seen (3) and (4) lead to a contradiction.

q.e.d.

Corollary 2. If $\mathcal{H}'_2 \neq \emptyset$ then, for every $h \in \mathcal{H} \setminus (\mathcal{H}_1 \cup \mathcal{H}_2)$, $p_\infty(h) = 0$.

Proof. The argument is straightforward by using for instance lemma 1 and corollary 1.

Now, let us consider the following sets:

$$\mathcal{H}_3 = \{h_1 \notin \mathcal{H}_1 \cup \mathcal{H}_2 / (\forall) h \notin \mathcal{H}_1 \cup \mathcal{H}_2, h \neq h_1, \lim_{t \rightarrow \infty} u_t(h)/u_t(h_1) = \infty\}$$

$$\mathcal{H}'_3 = \{h_1 \notin \mathcal{H}_1 \cup \mathcal{H}_2 / \lim_{s \rightarrow \infty} p_s(h_1) = 0\}.$$

Then, one can easily prove that the following relations hold:

Lemma 2.

$$a) \quad \mathcal{H}_3 = \mathcal{H}'_3$$

- b) $\mathcal{H}_2 \neq \emptyset$ if and only if $\mathcal{H}_3 \neq \emptyset$
 c) $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ if and only if $\mathcal{H}_2 \neq \emptyset$.

Now, the following theorem can be proved

Theorem 1. *The weighting process in the general sense is convergent to a degenerate distribution (i.e., $p_\infty(h) = 1$ for some $h \in \mathcal{H}$) if and only if $\mathcal{H}_2 \neq \emptyset$.*

Hence, we can suppose that $\mathcal{H}_2 = \emptyset$. According to lemma 2b we have $\mathcal{H}_3 = \emptyset$. Let us introduce the following set:

$$\mathcal{H}_4 = \{h_1 \notin \mathcal{H}_1 / (\forall) h \in \mathcal{H}_1, \lim_{t \rightarrow \infty} u_t(h)/u_t(h_1) = \lambda(h, h_1)\}$$

As a consequence of previous proved results, we have: $0 < \lambda(h, h_1) < \infty$ for any $h_1 \in \mathcal{H}_4$, $h \notin \mathcal{H}_1$.

Lemma 3. *If $\mathcal{H}_4 \neq \emptyset$, then $\mathcal{H}_4 = \mathcal{H} \setminus \mathcal{H}_1$.*

Proof. Obviously, we get $\mathcal{H}_4 \subseteq \mathcal{H} \setminus \mathcal{H}_1$. Now, let us suppose that $h_0 \in \mathcal{H} \setminus \mathcal{H}_1$, but $h_0 \notin \mathcal{H}_4$. Hence, there exists at least one element $h_1 \in \mathcal{H}_4$ such that:

$$(5) \quad \lim_{t \rightarrow \infty} u_t(h)/u_t(h_1) = \lambda(h, h_1),$$

for every $h \notin \mathcal{H}_1$. Since $h_0 \in \mathcal{H} \setminus \mathcal{H}_1$, we get (5) for $h = h_0$, i.e.,

$$(6) \quad \lim_{t \rightarrow \infty} u_t(h_0)/u_t(h_1) = \lambda(h_0, h_1).$$

From (5) and (6) we obtain

$$(7) \quad \lim_{t \rightarrow \infty} u_t(h)/u_t(h_0) = \lambda(h, h_1) / \lambda(h_0, h_1)$$

for every $h \notin \mathcal{H}_1$. But the relation (7) assesses that $h_0 \in \mathcal{H}_4$, which is a contradiction. This ends the proof.

q.e.d.

Lemma 4 *If $\mathcal{H}_2 = \emptyset$, then $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_4$*

Proof: By using lemma 3, the proof is straightforward. We are able now to prove the main result concerning our topic.

Theorem 2: *Let us suppose that $\mathcal{H}_2 = \emptyset$. Then, the weighting process in the general sense is convergent if and only if $\mathcal{H}_4 \neq \emptyset$.*

Proof. Let $\mathcal{H}_4 \neq \emptyset$ be. For $h \in \mathcal{H}_1$, we have $p_\infty(h) = 0$. According to lemma 4, since $\mathcal{H}_4 \neq \emptyset$, we get $\mathcal{H}_4 = \mathcal{H} \setminus \mathcal{H}_1$. We have to prove that for any $h_1 \in \mathcal{H}_4$, there exists the asymptotical probability, $p_\infty(h_1) = \lim_{t \rightarrow \infty} p_t(h_1)$. Let h_1 be an arbitrary element belonging to $\mathcal{H}_4$. Hence, we have

$$(8) \quad \lim_{t \rightarrow \infty} u_t(h)/u_t(h_1) = \lambda(h, h_1)$$

for any $h \notin \mathcal{H}_1$. But there exists a positive number t_0 such that for every $t > t_0$, we have

$$(9) \quad p_t(h_1) = p_0(h_1) \cdot \left[\sum_{\bar{h} \notin \mathcal{H}_1} p_0(\bar{h}) \cdot u_t(\bar{h})/u_t(h_1) \right]^{-1}.$$

By using (8) and (9) we obtain

$$(10) \quad \lim_{t \rightarrow \infty} p_t(h_1) = p_0(h_1) \cdot \left[\sum_{h \notin \mathcal{H}_1} p_0(h) \cdot \lambda(h, h_1) \right]^{-1}$$

i.e., there exists $p_\infty(h_1)$.

Conversely, we suppose that the weighting process is convergent, i.e., for any $h \in \mathcal{H}$ there exists the asymptotical probability $p_\infty(h) = \lim_{t \rightarrow \infty} p_t(h)$. According to lemma 1a we have $\mathcal{H}_1 = \mathcal{H}'_1$. By using theorem 1, since $\mathcal{H}_2 = \emptyset$, if the weighting process is convergent, its asymptotical distribution $(p_\infty(h))_h$ is not degenerate one. Hence, there are $h_1, h_2 \in \mathcal{H} \setminus \mathcal{H}_1$, $h_1 \neq h_2$ such that:

$$(11) \quad 0 < p_\infty(h_1) = \lim_{t \rightarrow \infty} p_t(h_1) < 1, \quad 0 < p_\infty(h_2) = \lim_{t \rightarrow \infty} p_t(h_2) < 1.$$

But there exists a positive number t_0 such that for any $t > t_0$, we have:

$$(12) \quad p_t(h_1) = p_0(h_1) \cdot u_t(h_1) \cdot \left[\sum_{h \notin \mathcal{H}_1} p_0(h) \cdot u_t(h) \right]^{-1}$$

$$(13) \quad p_t(h_2) = p_0(h_2) \cdot u_t(h_2) \cdot \left[\sum_{h \notin \mathcal{H}_1} p_0(h) \cdot u_t(h) \right]^{-1}$$

Hence, for any $t > t_0$, from (12) and (13) we get

$$(14) \quad p_t(h_1)/p_t(h_2) = p_0(h_1) \cdot u_t(h_1) / (p_0(h_2) \cdot u_t(h_2))$$

By using (11) and (14) we obtain

$$(15) \quad \lim_{t \rightarrow \infty} u_t(h_1)/u_t(h_2) = p_\infty(h_1) \cdot p_0(h_2) / (p_\infty(h_2) \cdot p_0(h_1))$$

Since for any $h_1 \notin \mathcal{H}_1$, we have $p_\infty(h_1) > 0$, by using lemma 1, and $\mathcal{H}_2 = \emptyset$, $\mathcal{H}'_3 = \emptyset$, the relation (15) holds for any $h_1 \notin \mathcal{H}_1$. Hence $h_2 \in \mathcal{H}_4$ and $\mathcal{H}_4 \neq \emptyset$. This completes the proof.

q.e.d.

Remark 1. $\mathcal{H}_4 \neq \emptyset$ if and only if the weighting process is convergent to a non-degenerate distribution $(p_n(h))_h$ such that $0 < p_\infty(h) < 1$ for any $h \in \mathcal{H} \setminus \mathcal{H}_1$.

Remark 2. If $(p_t(h))_{t, h}$ is a sequential weighting process then, $\mathcal{H}_1 = \{h \in \mathcal{H} \mid (\exists) t \in \mathbb{N}^*, \tilde{u}_t(h) = 0\}$.

Remark 3. If $\mathcal{H}_2 = \emptyset$ and the sequential weighting process is a convergent one then, $\lim_{t \rightarrow \infty} u_t(h_1)/u_t(h_2) = 1$, for any $h_1, h_2 \in \mathcal{H}_4$.

Remark 4. If $(p_t(h))_{t, h}$ is an iterated weighting process then $\mathcal{H}_1 = \emptyset$.

Remark 5. If $\mathcal{H}_2 = \emptyset$ then the iterated weighting process is a convergent weighting process if and only if for any $h_1, h_2 \in \mathcal{H}$ we have: $\lim_{t \rightarrow \infty} (u(h_1)/u(h_2))^t = \lambda(h_1, h_2)$, where $0 < \lambda(h_1, h_2) < 1$.

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AN ENTROPIC MEASURE OF THE UNILATERAL DEPENDENCE BETWEEN RANDOM VECTORS

ION PURCARU

In this paper the possibility to use the conditioned entropy as a measure of the unilateral dependence between random variables or vectors is presented.

1. General considerations

Let X be an arbitrary n -dimensional discrete random vector and $X = (Y_1, Y_2)$ a partition of X . Denote by $H(X)$, $H(Y_1)$ and $H(Y_2)$ the Shannon's entropies corresponding to the three random vectors. We know that the expression

$$(1) \quad W(X; Y_1, Y_2) = H(Y_1) + H(Y_2) - H(X) = W(Y_1, Y_2)$$

defines a measure of the interdependence between the random vectors Y_1 and Y_2 . This quantity represents the intensity of the cohesion between Y_1 and Y_2 as components of X (see [4], [7]—[9], [12]) and is named overall entropic correlation of the random vectors Y_1 and Y_2 (see [8], [9]).

As a result of the fulfilling properties of the quantity $W(Y_1, Y_2)$, a good measure of dependence between random variables or vectors is considered and starting from this measure we can define an entire family of entropic indicators of the interdependence named the entropic correlation family (see [8], [9]).

Although the entropic correlation $W(Y_1, Y_2)$ is a good measure of the interdependence it presents a drawback too.

The indicator measures the degree of the interdependence between Y_1 and Y_2 but it does not show in what measure Y_1 is dependent on Y_2 and in what measure Y_2 is dependent on Y_1 . By other words it measures the degree of the interdependence of the component parts but can not indicate the degree of the unilateral dependence between the component parts.

2. The conditioned entropy and the measuring of the unilateral dependence.

For the n -dimensional random vector $X = (Y_1, Y_2)$ let us consider Shannon's entropies $H(Y_1)$, $H(Y_2)$ and $H(X)$ corresponding to Y_1 , Y_2 and X

and denote by $H(Y_1/Y_2)$ and $H(Y_2/Y_1)$ the conditioned entropies corresponding to conditioned random vectors Y_1/Y_2 and Y_2/Y_1 (see [2], [8], [9]). We know that

$$(2) \quad H(Y_1, Y_2) = H(Y_1) + H(Y_2/Y_1) = H(Y_2) + H(Y_1/Y_2)$$

and we shall infer that $H(Y_1/Y_2) = 0$ if and only if $H(Y_1, Y_2) = H(Y_2)$. *Definition 1.* We shall say that the random vector Y_1 is completely dependent on Y_2 if $h(Y_1, Y_2) = h(Y_2)$.

In the relation (2) we observe that the equality $H(Y_1/Y_2) = 0$ does not imply the equality $H(Y_2/Y_1) = 0$ too. Therefore " Y_1 is completely dependent of Y_2 " does not imply " Y_2 is completely dependent of Y_1 ". This means that "the complete dependence" is an unilateral attribute for the random vector (Y_1, Y_2) .

In the relation (2) we infer the relation

$$(3) \quad H(Y_1) - H(Y_1/Y_2) = H(Y_2) - H(Y_2/Y_2)$$

named "informational balance" of the random vectors Y_1 and Y_2 . Also in relation (2) we deduce that the equality $H(Y_1/Y_2) = H(Y_1)$ is fulfilled if and only if $H(Y_1, Y_2) = H(Y_1) + H(Y_2)$, the last equality being fulfilled if and only if the random vectors Y_1 and Y_2 are probabilistically independent.

In the relation (3) we deduce that if $H(Y_1/Y_2) = H(Y_1)$ then $H(Y_2/Y_1) = H(Y_2)$ too. Therefore " Y_1 is independent of Y_2 " implies " Y_2 is independent of Y_1 " and as a result we can say that "independence" is a bilateral attribute of the random vectors Y_1 and Y_2 . As a result of these observations we can formulate the obvious theorem.

Theorem 1. The quantity $H(Y_1/Y_2)$ satisfies the followings

- 1) it is defined for any random vector (Y_1, Y_2)
- 2) it is not symmetrical namely $H(Y_1/Y_2) \neq H(Y_2/Y_1)$
- 3) $0 \leq H(Y_1/Y_2) \leq H(Y_1)$
- 4) $H(Y_1/Y_2) = 0$ if and only if Y_1 is completely dependent of Y_2
- 5) $H(Y_1/Y_2) = H(Y_1)$ if and only if the random vectors Y_1 and Y_2 are probabilistically independent
- 6) it is invariable for any biunivocal transformation of the random vectors Y_1 and Y_2 namely $H[\varphi_1(Y_1)/\varphi_2(Y_2)] = H(Y_1/Y_2)$ for any biunivocal functions φ_1 and $\varphi_2: R \rightarrow R$
- 7) $H(Y_1/Y_2) \leq H(Y_2/Y_1)$ if and only if $H(Y_1) \leq H(Y_2)$

If we divide the relation (2) by $H(Y_1, Y_2)$ we shall obtain the relation

$$(4) \quad \frac{H(Y_1)}{H(Y_1, Y_2)} + \frac{H(Y_2/Y_1)}{H(Y_1, Y_2)} = \frac{H(Y_2)}{H(Y_1, Y_2)} + \frac{H(Y_1/Y_2)}{H(Y_1, Y_2)}$$

and we shall say that each ratio represents the weight of the information quantity corresponding to the random vectors Y_1 , Y_2/Y_1 , Y_2 and Y_1/Y_2 in the information quantity of the couple (Y_1, Y_2) . As a result of this fact and of the property 7) in theorem 1 we can say that the dependence of Y_1 on Y_2 is greater than the dependence of Y_2 of Y_1 if the weight of the

information quantity of Y_1 in the information quantity of the couple (Y_1, Y_2) is smaller than that corresponding to the weight of the information quantity of Y_2 .

Definition 2. We shall say that the random vectors Y_1 and Y_2 have the same degree of unilateral dependence or that they are equally unilateral dependent if $H(Y_1/Y_2) = H(Y_2/Y_1)$.

In the relation (4) it results that the random vectors Y_1 and Y_2 are equally unilateral dependent if their weights of the information quantities in the information quantity of the couple (Y_1, Y_2) are equal.

Definition 3. We shall say that the random vectors Y_1 and Y_2 are mutually complete dependent if $H(Y_1/Y_2) = H(Y_2/Y_1)$.

Keeping into account the theorem 1 and the other remarks we can consider the conditioned entropy $H(Y_1/Y_2)$ as a measure of unilateral dependence between Y_1 and Y_2 .

In the relation (2) we can deduce that $H(Y_1/Y_2)$ may be interpreted as "informational distance" between Y_1 and (Y_1, Y_2) too. As a result of this fact it may be proved that the expression $d(Y_1, Y_2) = H(Y_1/Y_2) + H(Y_2/Y_1)$ is a metric in the set of the discrete random variables, called the Shannon's metric (see [5]).

Keeping into account the entropy properties we can immediately deduce :
Theorem 2. Between the indicators of overall and unilateral dependence there are always the relations

$$(5) \quad H(Y_i) = H(Y_i/Y_j) + W(Y_1, Y_2) \quad i, j = 1, 2 \quad i \neq j$$

$$(6) \quad H(Y_1/Y_2) + H(Y_2/Y_1) = H(Y_1, Y_2) - W(Y_1, Y_2)$$

As a result of the properties of the overall entropic correlation we can immediately infer the properties of the unilateral measure of dependence too, in theorem 1 presented above. At the same time the overall entropic correlation $W(Y_1, Y_2)$ as difference between the information quantity contained by Y_i when it is independent of Y_j and it is conditioned by Y_j , $1 < i \neq j < 2$, can be interpreted.

3. The practical necessity of the measuring of the unilateral dependence.

If the random vectors Y_1 and Y_2 are independent or if they are mutually complete dependent the problem is clear and none question exist.

But if neither of these situations are present then we can ask "what is the degree of dependence between the vectors Y_1 and Y_2 ?". The following numerical example illustrates the practical necessity of the measuring of the unilateral dependence.

Let $X = (X_1, X_2, X_3, X_4)$ be a random discrete vector, each component parts of X having two values, with the probabilistic joint distribution :

$p_{1111} = 3/16$	$p_{1211} = 0$	$p_{2111} = 0$	$p_{2211} = 2/16$
$p_{1112} = 3/16$	$p_{1212} = 0$	$p_{2112} = 0$	$p_{2212} = 2/16$
$p_{1121} = 1/16$	$p_{1221} = 0$	$p_{2121} = 0$	$p_{2221} = 2/16$
$p_{1122} = 1/16$	$p_{1222} = 0$	$p_{2122} = 0$	$p_{2222} = 2/16$

Performing the necessary computation we shall obtain the following entropies :

$$\begin{aligned}
 H(X_1, X_2, X_3, X_4) &= (28 - 3 \log 3)/8 \\
 H(X_1, X_2, X_3) &= (20 - 3 \log 3)/8 \\
 H(X_1, X_2, X_4) &= 2 \\
 H(X_1, X_3, X_4) &= H(X_2, X_3, X_4) = (28 - 3 \log 3)/8 \\
 H(X_1, X_2) &= H(X_1) = H(X_2) = H(X_4) = 1 \\
 H(X_1, X_3) &= H(X_2, X_3) = (20 - 3 \log 3)/8 \\
 H(X_1, X_4) &= H(X_2, X_4) = 2 \\
 H(X_3, X_4) &= (32 - 3 \log 3 - 5 \log 5)/8 \\
 H(X_3) &= (24 - 3 \log 3 - 5 \log 5)/8.
 \end{aligned}$$

and as a result we shall deduct that :

- 1) $W[X_1, X_2] = 1 = H(X_1) = H(X_2) = H(X_1, X_2) \Rightarrow H(X_1/X_2) = H(X_2/X_1) = 0$, therefore X_1 and X_2 are mutually complete dependent.
- 2) $W[X_2, X_4] = 0 \Rightarrow H(X_3, X_4) = H(X_3) + H(X_4) \Rightarrow H(X_3/X_4) = H(X_3)$ and $H(X_4/X_3) = H(X_4)$, therefore X_3 and X_4 are independent.
- 3) $W[X_1, X_3] = (12 - 5 \log 5)/8 \Rightarrow H(X_1/H_3) = (5 \log 5 - 4)/8$ and $H(X_3/X_1) = (12 - 3 \log 3)/8$ and because $H(X_1/X_3) \approx H(X_3/X_1)$ it results that the degree of unilateral dependence is the same approximately for X_1 and X_2 .
- 4) $W[X_1, (X_2, X_3)] = 1 = H(X_1) \neq H(X_2, X_3) \Rightarrow H[X_1/(X_2, X_3)] = 0$ and $H[(X_2, X_3)/X_1] = (12 - 3 \log 3)/8$, therefore X_1 is completely dependent of the vector (X_2, X_3) but reciprocally not.
- 5) $W[X_2, (X_1, X_4)] = 1 = H(X_2) \neq H(X_1, X_4) \Rightarrow H[X_2/(X_1, X_4)] = 0$ and $H[(X_1, X_4)/X_2] = 1$, therefore X_2 is completely dependent on the vector (X_1, X_4) but reciprocally not.
- 6) $W[X_4, (X_1, X_2)] = 0 \Rightarrow H[X_4/(X_1, X_2)] = H(X_4)$ and $H[(X_1, X_2)/X_4] = H(X_1, X_2)$ and it results that the variable X_4 and the vector (X_1, X_2) are independent.
- 7) $W[X_1, (X_2, X_3, X_4)] = 1 = H(X_1) \neq H(X_2, X_3, X_4) \Rightarrow H[X_1/(X_2, X_3, X_4)] = 0$ and $H[(X_2, X_3, X_4)/X_1] = (20 - 3 \log 3)/8$, therefore X_1 is completely dependent of the vector (X_2, X_3, X_4) but reciprocally not.
- 8) $W[X_3, (X_1, X_2, X_4)] = (12 - 5 \log 5)/8 \Rightarrow$ there is a certain interdependence between X_3 and (X_1, X_2, X_4) but because $H[X_3/(X_1, X_2, X_4)] = 1 < H[(X_1, X_2, X_4)/X_3] = (4 + 5 \log 5)/8$ it results that X_3 is dependent of (X_1, X_2, X_4) moreover than (X_1, X_2, X_4) is dependent of X_3 .
- 9) $W[X_4, (X_1, X_2, X_3)] = 0 \Rightarrow H[X_4/(X_1, X_2, X_3)] = H(X_4)$ and $H[(X_1, X_2, X_3)/X_4] = H(X_1, X_2, X_3)$, therefore X_4 and (X_1, X_2, X_3) are independent.
- 10) $W[(X_1, X_2), (X_3, X_4)] = (12 - 5 \log 5)/8 \Rightarrow$ there is a certain interdependence between the vectors (X_1, X_2) and (X_3, X_4) but because $H[(X_1, X_2)/(X_3, X_4)] = (5 \log 5 - 4)/8 < H[(X_3, X_4)/(X_1, X_2)] = (20 - 3 \log 3)/8$ it results that the vector (X_1, X_2) is dependent on the vector (X_3, X_4) moreover than (X_3, X_4) is dependent on (X_1, X_2) .

The presented numerical situations are sufficiently for justifying the practical necessity of the utilisation the unilateral measure of dependence.

We mention that for a complete examination of dependence between the random variables or random vectors taken together, the overall indicators of dependence and the unilateral measure of dependence are necessary indeed.

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SOLUTIONS OF BILOCAL POLYNOMIAL PROBLEMS BY LINEARIZATION

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In this paper it is given a constructive method for the analytic solutions of the bilocal polynomial problem, based on a linearization mapping, introduced and studied in [6], [7]. Development over the whole considered interval of the solution is obtained, as well as separation and development of all analytic solution when uniqueness does not hold.

Introduction

The bilocal problem for second order differential operators appears in many concrete models of mathematical physics [1], elasticity and plasticity [2], mechanics of fluids, etc.

In the nonlinear case, the coincidence degree results are applied to obtain easily verified existence conditions [4]

From the numerical point of view, collocation [3], shooting [5] and finite element method were used to get numerical approximations of the solution.

The aim of this paper is to give a constructive existence frame for the analytic solutions of the bilocal polynomial problem by means of a linearization method for polynomial operators ([6], [7]).

1. Let $\mathcal{X}, \mathcal{Y}$ be real Banach spaces.

Definition. [8] $P: \mathcal{X} \rightarrow \mathcal{Y}$ is a n -th degree polynomial operator if P may be represented as a sum:

$$(1.1) \quad Px = \sum_{h=0}^n L_h x^h$$

where $L_h: \mathcal{X}^h \rightarrow \mathcal{Y}$, $1 \leq h \leq n$, are h -linear operators [9], L_0 being the "constant" operator.

According to this definition, ordinary differential equations are polynomials of the unknown functions and their derivatives, with variable coefficients.

For first order polynomial systems:

$$(1.2) \quad y' = P(x, y) \quad x \in [a, b] \subset \mathbb{R}$$

where $y = (y_j)_{1 \leq j \leq n} \in [C^1([a, b])]^n$, $P(x, y) = (P_j(x, y))_{1 \leq j \leq n}$ and

$$(1.2') \quad P_j(x, y) = \sum_{|v| \leq p_j} a_j^v(x) y^v \quad a_j^v: [a, b] \rightarrow R$$

it was introduced in [4] an exponential transform of real parameters ξ_j , $1 \leq j \leq n$, that maps (1.2) into:

$$(1.3) \quad \mathcal{L}v \equiv \frac{\partial v}{\partial x} - \sum_{j=1}^n \xi_j P_j(x, D) v = 0$$

where $P_j(x, D)$ is the differential with respect to ξ_j polynomial, corresponding to (1.2).

Adding the initial conditions:

$$(1.4) \quad y_j(a) = y_j^0 \quad 1 \leq j \leq n$$

the linearization mapping $v(x, \xi)$ establishes an one-to-one correspondence between the set of solutions of the initial polynomial problem (1.2), (1.4), and the set of analytic with respect to $\xi = (\xi_j) \in R^n$ solutions of equation (1.3) under transformed initial conditions:

$$(1.5) \quad v(a, \xi) = \exp \left(\sum_{j=1}^n y_j^0 \xi_j \right)$$

provided $a_j^v(x)$ are bounded on $[a, b]$.

This is the core of the linearization method for polynomial operators, introduced and developed in [6], [7].

Consequently, the linear equation (1.3) fully represents the polynomial operator (1.2). Let us call (1.3) the *linearized form* of (1.2).

Define now the Banach spaces $\mathcal{B}_n$ and $\mathcal{C}_n$ as follows:

$$(1.6) \quad \mathcal{B}_n = \left\{ v: [a, b] \times R^n \rightarrow R \mid v(x, \xi) = \right.$$

$$\left. \sum_{|\gamma| \geq 0} v_\gamma(x) \xi^\gamma, \sup_{x \in [a, b]} |v_\gamma(x)| \leq \frac{M^{|\gamma|}}{\gamma!} \cdot K \right\}$$

$$\mathcal{C}_n = \left\{ v \in \mathcal{B}_n \mid \frac{\partial v}{\partial x} \in \mathcal{B}_n \right\}$$

Lemma 1. *The set of the linearized forms of the polynomial operators (1.2) is affine and maps $\mathcal{C}_n$ into $\mathcal{B}_n$.*

Proof. Obviously, since for any linear combination of every two linearized forms $\mathcal{L}$, $\mathcal{L}'$, with coefficients c_1 , c_2 satisfying $c_1 + c_2 = 1$:

$$(1.7) \quad c_1 \mathcal{L} v + c_2 \mathcal{L}' v \equiv (c_1 + c_2) \frac{\partial v}{\partial x} - \sum_{j=1}^n \xi_j [c_1 P_j(x, D) + c_2 Q_j(x, D)] v = \frac{\partial v}{\partial x} - \sum_{j=1}^n \xi_j [c_1 P_j(x, D) + c_2 Q_j(x, D)] v$$

hence (1.7) is the linearized form of the polynomial operator :

$$(1.8) \quad y'_j = c_1 P(x, y) + c_2 Q(x, y) \quad 1 \leq j \leq n$$

For the second part of the lemma, it is sufficient to prove that every differential with respect to ξ polynomial invariants $\mathcal{B}_n$. Let $P(x, D)$ be such an operator, with bounded on $[a, b]$ coefficients a_v . Then $v \in \mathcal{B}_n$ yields :

$$(1.9) \quad |P(x, D)v| = \left| \sum_{|\gamma| \leq p} a_\gamma(x) \sum_{|\gamma|} v_\gamma(x) A_{\gamma+\nu}^\nu \xi^\nu \right| \leq \\ \leq c \sum_{s \leq p} M^s \exp \left(M \sum_{j=1}^n |\xi_j| \right)$$

where

$$A_\gamma^\nu = \prod_{j=1}^n (\gamma_j - s_j)$$

That is, $P(x, D)$ allows a development :

$$(1.10) \quad \sum_{|\gamma|} b_\gamma(x) \xi^\gamma$$

$b_\gamma(x)$ being defined by :

$$(1.11) \quad b_\gamma(x) = \sum_{|\nu|} a_\nu(x) v_{\gamma+\nu}(x) A_{\gamma+\nu}^\nu$$

and obviously satisfying the inequality :

$$(1.12) \quad |b_\gamma(x)| \leq c \sum_{s \leq p} M^s \frac{M^{|\gamma|}}{\gamma}$$

which means that $P(x, D)v$ belongs to $\mathcal{B}_n$. Q.E.D.

Now, as to get back to the nonlinear problem there are needed only analytic with respect to ξ solutions of (1.3), let v be developed in a series :

$$(1.13) \quad v(x, \xi) = \sum_{|\gamma| \geq 0} v_\gamma(x) \xi^\gamma \quad x \in [a, b], \quad \xi \in R^n$$

Using (1.13) and (1.3) it is immediately obtained an infinite system of linear ordinary differential equations :

$$(1.14) \quad LV \equiv \frac{dV}{dx} - T(x)V = 0$$

for

$$V(x) = (v_\gamma(x))_{|\gamma| \in N} \quad v_{(0)} = 1$$

and $T(x)$ is a row — and column — finite matrix.

The linear system (1.14) must be solved under the conditions:

$$(1.15) \quad V(a) = \left(\frac{(y^0)^\gamma}{\gamma!} \right)_{|\gamma| \in N}$$

The general theory for infinite systems [10] gives no satisfactory results for the correspondence with the nonlinear initial problem (1.2), (1.4).

Consequently, special properties of the system (1.14) must be used to get back to the nonlinear problem, and this is done in

Proposition 1. Let $a_j(x)$ be polynomials:

$$(1.16) \quad a_j(x) = \sum_{0 \leq q_j \leq q_j} a_{j,q}^\gamma x^q$$

Then the linear and explicit algorithm:

$$(1.17) \quad (i+1) v_\gamma^{i+1} = \sum_{j=1}^n \sum_{|\nu| \leq p_j} \sum_{m=0}^{q_j} a_{j,m}^\nu v_{\gamma-\nu}^{i-m} A_{\gamma+\nu-e_j}^\nu$$

where $e_j = (\delta_j^s)_{1 \leq s \leq n}$, δ_j^s is the Kronecker delta,

$$(1.18) \quad v_\gamma^0 = \frac{(y^0)^\gamma}{\gamma!}; \quad v_{(0)}^m = 0; \quad v_{(0)}^0 = 1 \quad m \in N$$

i) converges for $|x - a| < 1/cpQ$, $p = \max_{1 \leq j \leq n} \{p_j\}$,

$$c = \max_{\nu, q, j} |a_{j,q}^\nu|, \quad Q = \sum_{j=1}^n \sum_{|\nu| \leq p_j} (y^0)^\nu$$

ii) $\{v_{e_j}^m (x - a)^m\}_{1 \leq j \leq n}$ represents the Taylor series development for the solution of the initial polynomial problem (1.2), (1.4).

Proof. It is easily noticed that for the first step:

$$\begin{aligned} (1.19) \quad |v_\gamma^1| &\leq c \sum_{j=1}^n \sum_{|\nu| \leq p_j} \frac{|(y^0)^{\gamma+\nu}|}{(\gamma+\nu)!} A_{\gamma+\nu-e_j}^\nu = \\ &= c \sum_{j=1}^n \sum_{|\nu| \leq p_j} \frac{|(y^0)^{\gamma+\nu}|}{(\gamma+\nu)!} \frac{(\gamma+\nu-e_j)!}{(\gamma-e_j)!} = \\ &= c \sum_{j=1}^n \sum_{|\nu| \leq p_j} \frac{|(y^0)^{\gamma+\nu}|}{(\gamma_j+\nu_j)(\gamma-e_j)!} \leq cQ \frac{|(y^0)^\gamma|}{\gamma!} \end{aligned}$$

and obviously, for an arbitrary step m :

$$(1.20) \quad |v_\gamma^m| \leq c \sum_{j=1}^n p_j \sum_{|\gamma| \leq p_j} \sum_{q=0}^{q_j} |v_{\gamma+\gamma}^{m-q}| |a|_{j,q}^\gamma \leq (cpQ)^m \frac{|(y^0)^\gamma|}{\gamma!}$$

As a consequence of the two last inequalities, for $|x - a| < \frac{1}{cpQ}$, the series:

$$(1.21) \quad v(x, \xi) = \sum_{|\gamma| \geq 0, m \geq 0} v_\gamma^m (x - a)^m \xi^\gamma$$

converges for every ξ and satisfies the inequality:

$$(1.22) \quad |v(x, \xi)| \leq \frac{1}{1 - (x - a) cpQ} \sum_{|\gamma| \geq 0} |\xi|^{|\gamma|} \frac{|(y^0)^\gamma|}{\gamma!} = \frac{\exp \left(\sum_{j=1}^n |\xi_j| |y_j^0| \right)}{1 - (x - a) cpQ}$$

and, by the previous remarks, v still keeps its exponential form $\exp \left(\sum_{j=1}^n (\xi_j h_j) \right)$. Therefore, $v_\gamma(x) = \sum_m v_\gamma^m (x - a)^m$ are in fact $y^\gamma / \gamma!$, that completes the proof. of ii). Q.E.D.

Remark 1. The algorithm (1.17), (1.18) can be written in a more suggestive way, using (1.14). Indeed, the general form of the solution $V(x)$ of (1.14) under the conditions (1.15) is

$$(1.23) \quad V(x) = \prod (x) V(a)$$

where

$$(1.24) \quad V(a) = \{v_\gamma(a) | \gamma \in N\} = \left(\frac{(y_0)^\gamma}{\gamma!} \right)_{|\gamma| \in N}$$

and $\prod (x)$ is given by:

$$(1.25) \quad \prod (x) = \sum_{m \geq 0} T_m(a) (x - a)^m / m!$$

with $T_0 = I$ — the unit matrix and $T_m(a)$, row — and column — finite matrices obtained from the recurrence formula:

$$(1.26) \quad T_m(a) = \frac{dT_{m-1}}{dx} \Big|_{x=a} + T_{m-1} T \Big|_{x=a}, \quad T_1 = T(a)$$

As it is seen, in determining $\prod (x)$, only the coefficients of the polynomial system (1.2) are involved, and none of the initial conditions. To

get the solution of the polynomial system (1.2) only the first n rows of $\Pi(x)$ are needed. (see proposition 1, ii)).

Therefore, proposition 1 points out that the linearization method for polynomial operators gets the Taylor series development of the solution of (1.2), (1.4) up to an arbitrary order and besides, by (1.24), it gives a local representation for the general solution of the polynomial operator (1.2).

2. These results may be immediately used to the study of Picard's problem — the bilocal problem :

$$(2.1) \quad y'' = \sum_{j+k \leq p} a_{jk}(x) y^j (y')^k \quad a_{jk} : [a, b] \rightarrow R$$

$$(2.2) \quad y(a) = y_0; \quad y(b) = y_1$$

As an application of proposition 1 in this case, we prove

Proposition 2. Let $a_{jk}(x)$ be polynomials as in (1.16) and $\Pi(x)$ be the corresponding to (2.1) by linearization' matrix. Then :

i) the number of the analytic on $[a, b]$ solutions of the bilocal polynomial problem (2.1), (2.2) coincides with the number of the solutions of the functionale quation :

$$(2.3) \quad f(\beta) = y_1$$

where $f(\beta)$ is the first component of

$$(2.4) \quad \Pi(b) V_{\beta}(a)$$

with

$$(2.5) \quad V_{\beta}(a) = (v_{\gamma}(a))_{|\gamma| \in N} = \left(\frac{y_0^j \beta^k}{j! k!} \right)_{j+k \in N}$$

if $f(\beta)$ is consistent.

ii) the corresponding solutions of (2.1), (2.2) are first components of

$$(2.6) \quad \Pi(x) V_{\bar{\beta}}(a)$$

for every $\bar{\beta}$ satisfying (2.3).

Proof. By proposition 1 and remark 1, the general solution of (2.1) satisfying the initial conditions

$$(2.7) \quad y(a) = 0, \quad y'(a) = \beta$$

is the first component of $\Pi(x) V_{\beta}(a)$. An analytic on $[a, b]$ solution of (2.1), (2.2) must then obviously satisfy (2.3). Now, replace β by a solution $\bar{\beta}$ of (2.3). Then (2.6) gives a solution of the bilocal polynomial problem and since β must necessarily satisfy (2.3), there are no other analytic solutions of (2.1), (2.2). Q.E.D.

Remark 2. The consistency of $f(\beta)$ is not ensured by proposition 2. However, from proposition 2, ii), it follows that an admissible length for $[a, b]$ in the bilocal problem is:

$$(2.8) \quad l = |b - a| < \frac{1}{cpQ}$$

But it will be shown (see section 3) that this is not necessarily the best choice for l .

In the case of constant coefficients a_{jk} , the expression of the matrix $\Pi(x)$ is simplified and thus the following result is a natural consequence of proposition 2:

Corollary 1. The analytic on $[a, b]$ solutions of the bilocal polynomial problem (2.1), (2.2) with constant a_{jk} are given by

$$(2.9) \quad y(x) = [\exp(T(x - a)) V_{\beta}(a)]_1$$

with $f(\beta)$ satisfying

$$(2.10) \quad f(\beta) = y_1, f(\beta) = [\exp(T(b - a)) V(a)]_1$$

Proof. Applying proposition 2, notice that in the case of constant a_{jk} , $\Pi(x)$ is given by:

$$(2.11) \quad \exp(T(x - a)) = \sum_{m \geq 0} T^m \frac{(x - a)^m}{m!}$$

where T^m , $m \geq 0$, are obviously row- and column- finite.

Remark 3. Proposition 2 points out two special possibilities of the linearization method, applied to bilocal problems, that cannot be found in classic methods. Firstly, linearization method gives the approximation of analytic solutions over the whole considered interval $[a, b]$, the accuracy increasing with the order of approximation. Secondly, it separates and gives all analytic solutions, even when uniqueness does not hold.

To justify the approximation of $f(\beta)$ by the truncated $f^m(\beta)$, let us define on $\mathcal{C}_n$ the projectors $\mathcal{P}_m$ that keep in (1.13) only those $v^{\gamma}(x)$ with $|\gamma| \leq m$. Obviously, $\mathcal{P}_m \mathcal{C}_n \subset \mathcal{C}_n$, and $\ker \mathcal{L}$, $\ker \mathcal{L} \mathcal{P}_m$ are closed Banach subspaces of $\mathcal{C}_n$ ($\ker$ is the usual notation for the kernel).

Proposition 3. For every linearized form $\mathcal{L}$ of a polynomial system (1.2):

$$(2.12) \quad \bigcap_{m \in N} \ker \mathcal{L} \mathcal{P}_m = \ker \mathcal{L}$$

Proof: Inclusion $\supset$ is obvious. Conversely, let $v \in \ker \mathcal{L} \mathcal{P}_m$ $m \in N$. Then, by [11], $\sup_{|x \in a, b|} |\mathcal{L}(\mathcal{P}_m v - v)| \leq \varepsilon$ for every $m > N(\varepsilon)$, that means

$\mathcal{L}v = 0$. Q.E.D.

Of course, (2.12) still holds for $\ker L$, $\ker L \mathcal{P}_m$.

3. *Example.* Consider the bilocal polynomial problem :

$$(3.1) \quad \begin{aligned} G(y) &\equiv y'' + yy' + y^3 = 0 \\ y(0) &= y(\pi) = 0 \end{aligned}$$

whose analytic on $[0, \pi]$ solutions are :

$$y_1 = 0; \quad y_2 = \sin x; \quad y_3 = -\sin x$$

Introducing the linearization mapping $v(x, \sigma, \xi)$, $\sigma, \xi \in R$, the linearized form of (3.1) is :

$$(3.2) \quad \frac{\partial v}{\partial x} = \sigma \frac{\partial v}{\partial \xi} - \xi \left(\frac{\partial^3 v}{\partial \sigma^3} + \frac{\partial^3 v}{\partial \sigma \partial \xi^2} \right)$$

with the associated linear system :

$$(3.3) \quad \begin{aligned} \frac{dv_{j,k}}{dx} &= (k+1)v_{j-1, k+1} - (j+3)(j+2)(j+1)v_{j+3, k-1} - \\ &\quad - k(k+1)(j+1)v_{j+1, k+1} \quad j, k \in N \cup \{0\} \end{aligned}$$

The constant row- and column- finite matrix T , corresponding to (3.3) is of the form :

$$(3.4) \quad T = \begin{Bmatrix} S_1 & 0 & S_{11} & 0 & 0 & 0 & \cdot & \cdot \\ 0 & S_2 & 0 & S_{22} & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Bmatrix}$$

where :

$$S_i = (j \delta_k^{j+k})_{j, k \leq i+1}$$

$$S_{ii} = (-A_{i+3=k}^3 \delta_{k+1}^i - (i+1-k)k(k+1) \delta_k^{i+1} + 6(i-1) \delta_j^i)_{\substack{j \leq i+3 \\ k \leq i+1}}$$

Since in this case $c = 1$, $p = 3$, $Q = 2$, and consequently $\pi > 1/cpQ$ (see proposition 1), hence the conditions for admissible length can be improved, as it was already mentioned (remark 2).

Now, by proposition 3 and corollary 1, the first component of the approximating sums :

$$\Pi_m(\pi) = \sum_{k=0}^m T^k \frac{\pi^k}{k!}$$

gives for increasing m better and better approximations $f^m(\beta)$ for $f(\beta)$, obtained as first components of $\Pi_m V_\beta(0)$. The roots of $f^m(\beta)$, as well

as the corresponding approximations for the solutions, are enlisted as follows :

m	$f^m(\beta)$	β_j^m	$v_{10}^m = y^m(x)$
2	$\pi\beta$	$\beta_1^2 = 0$	$y_1^2 = 0$
3	$\pi\beta \left(1 - \frac{\pi^2\beta^2}{6}\right)$	$\beta_1^3 = 0$ $\beta_2^3 = \pm 0.79$	$y_1^3 = 0$ $y_2^3 = 0.79x - 0.0808x^3$ $y_3^3 = -0.79x + 0.0808x^3$
4	IDEM		
5	$\pi\beta \left[1 - \frac{\beta^2}{6} \left(\pi^2 + \frac{3\pi^2}{10}\right) + \frac{7\pi^4\beta^4}{120}\right]$	$\beta_1^5 = 0$ $\beta_{2,3}^5 = \pm 0.981$ $\beta_{4,5}^5 = \pm 0.435$	$y_1^5 = 0$ $y = 0.981x - 0.155x^3 + 0.0039x^5$ $y_2^5 = -0.981x + 0.155x^3 - 0.0039x^5$
6	IDEM		
	$\pi\beta \left[1 - \frac{\beta^2}{6} \left(\pi^2 + \frac{3\pi^4}{10}\right) + \beta^4 \left(\frac{7\pi^4}{120} + \frac{\pi^6}{42}\right) - \frac{127}{5040} \pi^6 \beta\right]$	$\beta_1^7 = 0$ $\beta_{2,3}^7 = \pm 0.9899$ $(\beta_j^7)^2 \text{ complex } 1 \leq j \leq 4$	$y_1^7 = 0$ $y_2^7 = 0.989x - 0.16166x^3 + 0.0069x + 0.00056x^7$ $y = -0.989x + 0.16166x^3 - 0.0069x - 0.00056x^7$

This table points out three distinct sequences (β_j^m) , $1 \leq j \leq 3$, converging with m to $y_j'(0)$ respectively. The other roots of $f^m(\beta)$ are not significant, as it is seen. The corresponding $y_j^m(x)$ tend with m to $y_j(x)$, $1 < j < 3$, approximating them over the whole interval $[0, \pi)$.

Remark 4. $f^m(\beta)$ are all odd, as well as the solutions y_j and the operator G .

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DÉTERMINATION D'UN INTERVALLE CONTENANT LA SOLUTION DE L'ÉQUATION POUR LE PRINCIPE DE L'ENTROPIE MAXIMALE

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Soit, f , une variable aléatoire, prenant n valeurs, $n < \infty$, $f_1, f_2, \dots, f_n$, et soit $\langle f \rangle$ l'esperance mathématique de cette variable.

Pour déterminer la distribution de probabilité, $\{p_k | k = 1, 2, \dots, n\}$ assujettie à la contrainte :

$\sum_{k=1}^n p_k \cdot f_k = \langle f \rangle$, en appliquant le principe de maximisation de l'entropie, il faut résoudre l'équation :

$$(1) \quad F(\beta) = 0, \text{ où}$$

$$(2) \quad F(\beta) = \sum_{k=1}^n (\langle f \rangle - f_k) e^{-\beta f_k} \text{ et}$$

β est le multiplicateur de Lagrange.

L'équation (1) a les mêmes solutions que l'équation :

$$(3) \quad G(\beta) = 0, \text{ où}$$

$$(4) \quad G(\beta) = \sum_{k=1}^n (\langle f \rangle - f_k) e^{\beta(\langle f \rangle - f_k)}$$

Or, on sait que la fonction G , c'est une fonction croissant sur R et que l'équation (3) à une seule solution.

Generalment, l'équation (3) n'a pas une solution exacte. Certaines valeurs particulières pour n et f_k , $k = 1, 2, \dots, n$, permettent la détermination d'une solution exacte de l'équation (3).

Exemple 1. $n = 3$, $f_1 = f_2 - c$; $f_3 = f_2 + c$, $c > 0$; on obtient

$$(5) \quad (\langle f \rangle - f_2 + c)x^2 + (\langle f \rangle - f_2)x + (\langle f \rangle - f_2 - c) = 0, \text{ où } x = e^{\beta c}$$

L'équation (5) a toujours deux solutions x_1, x_2 réelles et distinctes, pour $f_2 \neq \langle f \rangle$ et $x_1, x_2 < 0$. Soit $x_1 > 0$, alors β^* , la solution exacte de l'équation (5) est $\beta^* = \frac{1}{c} \ln x_1$.

Analogiquement, pour $n = 4$ et f prenant des valeurs équidistantes, on obtient une équation algébrique de troisième degré qui a une seule racine positive.

Exemple 2. Soit $\bar{f} = \frac{1}{n} \sum_{k=1}^n f_k$. Si $\bar{f} = \langle f \rangle$, alors, l'équation (3) a la solution exacte $\beta^* = 0$.

Pour déterminer la solution approchée de l'équation (3), il faut connaître un intervalle contenant la solution β^* . Supposons que :

$$(6) \quad f_1 < f_2 < \dots < f_r \leq \langle f \rangle < f_{r+1} < \dots < f_n$$

Notons que

$$(7) \quad G(0) = n \langle f \rangle - n \bar{f} = n(\langle f \rangle - \bar{f})$$

La formule (7) entraîne :

- a) si $\langle f \rangle = \bar{f}$, alors $G(0) = 0$ et $\beta^* = 0$ (l'exemple 2);
- b) si $\langle f \rangle > \bar{f}$, alors $G(0) > 0$ et $\beta^* < 0$, en tenant compte de monotonie de G , et
- c) si $\langle f \rangle < \bar{f}$, alors $G(0) < 0$ et $\beta^* > 0$.

A. Pour le cas b), cherchons une valeur $\beta_0 < 0$, de telle sorte que $\beta_0 < \beta^* < 0$.

Soit

$$(8) \quad \beta < 0 \text{ de telle sorte que}$$

$$(9) \quad G(\beta) \leq 0, \text{ c'est-à-dire}$$

$$(10) \quad (\langle f \rangle - f_1) e^{\beta(\langle f \rangle - f_1)} + \dots + (\langle f \rangle - f_r) e^{\beta(\langle f \rangle - f_r)} \leq \\ \leq (f_{r+1} - \langle f \rangle) e^{\beta(\langle f \rangle - f_{r+1})} + \dots + (f_n - \langle f \rangle) e^{\beta(\langle f \rangle - f_n)}$$

Les relations (6) et (8) conduisent à

$$(11) \quad \begin{cases} e^{\beta(\langle f \rangle - f_r)} > e^{\beta(\langle f \rangle - f_i)} & \text{pour } i = 1, 2, \dots, r-1 \text{ et} \\ e^{\beta(\langle f \rangle - f_{r+1})} < e^{\beta(\langle f \rangle - f_j)} & \text{pour } j = r+2, \dots, n \end{cases}$$

Si le côté droit de l'inégalité (10) est minoré et le côté gauche est majoré en tenant compte des relations (11), on a :

$$(12) \quad \left(r \langle f \rangle - \sum_{k=1}^r f_k \right) e^{\beta(\langle f \rangle - f_r)} \leq \left(\sum_{k=r+1}^n f_k - (n-r) \langle f \rangle \right) e^{\beta(\langle f \rangle - f_{r+1})}$$

En multipliant l'inégalité (12) par $e^{-\beta(\langle f \rangle - f_r)}$ en passant aux logarithmes et en tenant compte de la relation (6), on obtient :

$$(13) \quad \beta \leq \frac{1}{f_r - f_{r+1}} \ln \left[\frac{r \langle f \rangle - \sum_{k=1}^r f_k}{\sum_{k=r+1}^n f_k - (n-r) \langle f \rangle} \right] = \beta_0$$

Montrons maintenant que le logarithme de la relation (13) est positif, et alors β_0 étant négative et vérifiant la relation (9), c'est la valeur cherchée. Donc :

$$\frac{r \langle f \rangle - \sum_{k=1}^r f_k}{\sum_{k=r+1}^n f_k - (n-r) \langle f \rangle} - 1 = \frac{r \langle f \rangle + (n-r) \langle f \rangle - \sum_{k=1}^n f_k}{\sum_{k=r+1}^n f_k - (n-r) \langle f \rangle} > 0$$

parce que $\langle f \rangle > \bar{f}$

B. Pour le cas c), cherchons une valeur $\beta_1 > 0$, de telle sorte que $0 < \beta^* < \beta_1$.

Soit

$$(14) \quad \beta > 0 \text{ de telle sorte que}$$

$$(15) \quad G(\beta) \geq 0, \text{ c'est-à-dire}$$

$$(16) \quad (\langle f \rangle - f_1) e^{\beta(\langle f \rangle - f_1)} + \dots + (\langle f \rangle - f_r) e^{\beta(\langle f \rangle - f_r)} \geq \\ \geq (f_{r+1} - \langle f \rangle) e^{\beta(\langle f \rangle - f_{r+1})} + \dots + (f_n - \langle f \rangle) e^{\beta(\langle f \rangle - f_n)}$$

Si le côté gauche de l'inégalité (16) est minoré et le côté droit est majoré en tenant compte des relations (11), on a

$$(17) \quad \left(r \langle f \rangle - \sum_{k=1}^r f_k \right) e^{\beta(\langle f \rangle - f_r)} \geq \left(\sum_{k=r+1}^n f_k - (n-r) \langle f \rangle \right) e^{\beta(\langle f \rangle - f_{r+1})}$$

En multipliant l'inégalité (17) par $e^{-\beta(\langle f \rangle - f_{r+1})}$, en passant aux logarithmes et tenant compte de la relation (6), on obtient :

$$(18) \quad \beta \geq \frac{1}{f_{r+1} - f_r} \ln \left[\frac{\sum_{k=r+1}^n f_k - (n-r) \langle f \rangle}{r \langle f \rangle - \sum_{k=1}^r f_k} \right] = \\ = \frac{1}{f_r - f_{r+1}} \ln \left[\frac{r \langle f \rangle - \sum_{k=1}^r f_k}{\sum_{k=r+1}^n f_k - (n-r) \langle f \rangle} \right] = \beta_1$$

Montrons maintenant que le dernier logarithme de la relation (18) est négatif et alors β_1 étant positive et vérifiant la relation (15), c'est la valeur cherchée. Donc :

$$\frac{r\langle f \rangle - \sum_{k=1}^r f_k}{\sum_{k=r+1}^n f_k - (n-r)\langle f \rangle} - 1 = \frac{r\langle f \rangle + (n-r)\langle f \rangle - \sum_{k=1}^n f_k}{\sum_{k=r+1}^n f_k - (n-r)\langle f \rangle} < 0$$

parce que $\langle f \rangle < \bar{f}$.

Remarque 1. La relation (6) entraîne :

$$(19) \quad \langle f \rangle - f_1 > \langle f \rangle - f_i, \text{ pour } i = 2, 3, \dots, r \text{ et } f_{r+1} - \langle f \rangle < f_j - \langle f \rangle \\ \text{pour } j = r + 2, \dots, n$$

$$(20) \quad \langle f \rangle - f_r < \langle f \rangle - f_i \text{ pour } i = 1, 2, \dots, r-1, \text{ et } f_n - \langle f \rangle > f_j - \langle f \rangle \\ \text{pour } j = r + 1, \dots, n-1.$$

D'une façon analogue, en employant les relations (19) et (20) à côté des relations (11), on obtient :

$$(21) \quad \beta_0^{(1)} = \frac{1}{f_{r+1} - f_r} \ln \left[\frac{(n-r)(f_{r+1} - \langle f \rangle)}{r(\langle f \rangle - f_1)} \right] \text{ et}$$

$$(22) \quad \beta_1^{(1)} = \frac{1}{f_{r+1} - f_r} \ln \left[\frac{(n-r)(f_n - \langle f \rangle)}{r(\langle f \rangle - f_r)} \right]$$

Remarque 2. Ecrivons la relation (9) sous la forme :

$$(23) \quad (\langle f \rangle - f_1) e^{\beta(\langle f \rangle - f_1)} + \dots + (\langle f \rangle - f_{n-1}) e^{\beta(\langle f \rangle - f_{n-1})} \leq (f_n - \langle f \rangle) e^{\beta(\langle f \rangle - f_n)}$$

En appliquant un raisonnement analogue de A à la relation (23), on obtient :

$$(24) \quad \beta_0^{(2)} = \frac{1}{f_{n-1} - f_n} \ln \left[\frac{(n-1)\langle f \rangle - \sum_{k=1}^{n-1} f_k}{f_n - \langle f \rangle} \right] \text{ et}$$

$$(25) \quad \beta_0^{(3)} = \frac{1}{f_{n-1} - f_n} \ln \left[\frac{(n-1)(\langle f \rangle - f_1)}{f_n - \langle f \rangle} \right].$$

En partant de la relation (15) écrite sous la forme :

$$(26) \quad (\langle f \rangle - f_1) e^{\beta(\langle f \rangle - f_1)} \geq (f_2 - \langle f \rangle) e^{\beta(\langle f \rangle - f_2)} + \dots + (f_n - \langle f \rangle) e^{\beta(\langle f \rangle - f_n)}$$

et appliquant un raisonnement analogue de B, on a :

$$(27) \quad \beta_1^{(2)} = \frac{1}{f_2 - f_1} \ln \left[\frac{\sum_{k=2}^n f_k - (n-1)\langle f \rangle}{\langle f \rangle - f_1} \right] \text{ et}$$

$$(28) \quad \beta_1^{(3)} = \frac{1}{f_2 - f_1} \ln \left[\frac{(n-1)(f_n - \langle f \rangle)}{\langle f \rangle - f_1} \right]$$

Notons que chaque valeur $\beta_0, \beta_0^{(i)}, \beta_1, \beta_1^{(i)}$ ($i = 1, 2, 3$) emploie d'une manière particulière toutes ou quelques valeurs de la variable aléatoire f .

Exemple numérique. Soit f ayant les valeurs :

$f_1 = -17; f_2 = -12; f_3 = -3; f_4 = 0; f_5 = 1; f_6 = 7; f_7 = 8$
et $\bar{f} = -2.29$.

Supposons que :

a) $\langle f \rangle = 11$

Alors $\langle f \rangle < \bar{f}$ et $\beta^* > 0$; on obtient :

$$\beta_1 = 0.25262 \quad \text{et} \quad G(\beta_1) = 25.93443$$

$$\beta_1^{(1)} = 0.42897 \quad G(\beta_1^{(1)}) = 79.79060$$

$$\beta_1^{(2)} = 0.48259 \quad G(\beta_1^{(2)}) = 109.91318$$

$$\beta_1^{(3)} = 0.58889 \quad G(\beta_1^{(3)}) = 207.12820$$

b). $\langle f \rangle = 6$

Alors $\langle f \rangle > \bar{f}$ et $\beta^* < 0$; on obtient :

$$\beta_0 = -0.50204 \quad \text{et} \quad G(\beta_0) = -6.30907$$

$$\beta_0^{(1)} = -0.675530 \quad G(\beta_0^{(1)}) = -9.38814$$

$$\beta_0^{(2)} = -3.40120 \quad G(\beta_0^{(2)}) = -1829.99902$$

$$\beta_0^{(3)} = -4.23411 \quad G(\beta_0^{(3)}) = -9590.99219$$

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GENERATIVE SYSTEMS WITH GROWTH FUNCTIONS BY POLYNOMIAL TYPE OR RECURSIVE TYPE

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I. Introductory notions

Let I be a non-empty, finite set, called the set of cells. An element $a \in I$ is called a cell.

A stage w is an application $w: \{1, 2, \dots, n\} \rightarrow I$. The number n represents the value (dimension) of the stage w and will be denoted by $|w|$ ($n = |w|$).

Example 1: Let $I = \{a, b, c\}$ and w be a stage with $|w| = 7$. The stage $w: \{1, 2, \dots, 6, 7\} \rightarrow I$ is thus defined: $w(1) = c$, $w(2) = b$, $w(3) = c$, $w(4) = a$, $w(5) = b$, $w(6) = a$, $w(7) = a$.

Observation 1. In the stage w we are not interested in the order of cells, but in their number (the value of the stage w) and in the components of the stage w only (how many types of different cells exist in w , and the number of cells of the same type).

Let us denote a stage w of value n by $w = w(1) \dots w(n)$, understanding by w any permutation $w(i_1) \dots w(i_n)$ of the n cells of the stage w .

By I^+ we denote the set of all stages over the set I of the cells.

Therefore, the stage from the example 1 $w = cbcabaa$ is identical with the stage $bbacaac$ or $aaabbcc$ or with any other combination of three cells of the a type, two of the b type and two of the c type.

Remark 1. If in a stage $w = a_1 a_2 \dots a_i \dots a_{n-1} a_n$ we want to refer to a cell of type a_i , then without restricting the generality of the problem, by considering $w = a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_{n-1} a_n a_i$, we could consider the cell from the n -th position (that is, the cell a_n); if we want to refer to two cells, then we could consider the cells $a_{n-1} a_n$; and so on.

Observation 2. We shall identify the stages of value one with the elements of the set I ; thus the application $w: \{1\} \rightarrow I$ given by $w(1) = i \in I$ will be simply denoted by i .

We shall add to the stages from I^+ the empty stage $\lambda: \emptyset \rightarrow I$ and by convention we decide the value of this stage to be 0, $|\lambda| = 0$.

Let us denote $I^* = I^+ \cup \{\lambda\}$. Intuitively, we could say that the set of cells of a stage evaporates (disappears); in this case the value of the stage becomes zero (actually there won't be any cell any more).

Definition 1. An $\mathcal{L}$ -system is a triplet $L = (I, b, S)$ where:

- I is the set of cells;
- $b \in I$, b representing the basic cell or the initial stage;
- S is a generative system $S = \{s_1, s_2, \dots, s_k\}$ where s_i , $1 \leq i \leq k$, are rules of the form $a \rightarrow w_a$ (where $a \in I$, $w_a \in I^*$); these rules are called successions ($w_a = \text{succ}(a)$).

Definition 2. The stage w_1 passes into the stage w_2 in the $\mathcal{L}$ -system $L = (I, b, S)$ if $w_1 = a_1 a_2, \dots, a_j$ and $w_2 = w_{a_1} w_{a_2} \dots w_{a_j}$ in which $a_1, a_2, \dots, a_j \in I$; $w_{a_1}, w_{a_2}, \dots, w_{a_j} \in I^*$ and $a_i \rightarrow w_{a_i} \in S$, $1 \leq i \leq j$. We shall write $w_1 \Rightarrow w_2$.

Definition 3. A growth of the $\mathcal{L}$ -system is a series of stages $w_1, w_2, \dots, \dots, w_m$ so that $w_j \Rightarrow w_{j+1}$, $1 \leq j \leq m - 1$. We shall say that this growth has m stages: w_1 -the initial stage or the first growth stage, w_2 -the second growth stage, $\dots$, w_m -the m -th growth stage or the present stage.

When $\{w_1, w_2, \dots, w_m\}$ represents a growth and we are interested in all stages $w_1, w_2, \dots, w_m$ we shall write $w_1 \Rightarrow w_2 \Rightarrow \dots \Rightarrow w_m$; in the case when we are interested only in the initial and the present stages

we shall write $w_1 \overset{+}{\Rightarrow} w_m$. We shall write $w_1 \overset{*}{\Rightarrow} w_m$ if $\begin{cases} w_1 = w_m (m=1) \\ \text{or} \\ w_1 \overset{+}{\Rightarrow} w_m \end{cases}$.

Example 2.

Let $L_1 = (\{a, b, c, d\}, b, S)$ $S = \begin{cases} a \rightarrow a \\ b \rightarrow ab \end{cases}$.

We consider the following growth, schematically represented in figure 1: $b \Rightarrow ab \Rightarrow aab \Rightarrow aaab \Rightarrow aaaab \Rightarrow aaaaaab$.

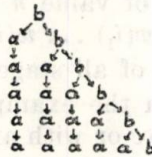


Fig. 1

Observation 3. The cells c and d do not appear in the generative system S , that's why we'll name them inactive cells. The cells a and b , i.e., the cells which appear in the generative system, will be called active cells.

Thus, in order to model the given $\mathcal{L}$ -system L_1 we need only the cells a and b and we could „simplify” the $\mathcal{L}$ -system L_1 , obtaining the $\mathcal{L}$ -system: $L = (\{a, b\}, b, S)$, $S = \begin{cases} a \rightarrow a \\ b \rightarrow ab \end{cases}$.

Generally, if an $\mathcal{L}$ -system is given $L = (I, b, S)$ we shall consider I representing the set of active cells.

Definition 4. The $\mathcal{L}$ -system $L = (I, b, S)$ is called *deterministic* if for any $a, a \in I$, there exists only one succession of the form $a \rightarrow w_a$. In this case the $\mathcal{L}$ -system has a unique growth.

The $\mathcal{L}$ -system from the second example is a deterministic one.

An active cell $a \in I$ can fulfill one of the following functions :

1. (C) *conservation* $|\text{succ}(a)| = 1$ and $a = \text{succ}(a)$
2. (M) *modification* $|\text{succ}(a)| = 1$ and $a \neq \text{succ}(a)$
3. (D) *division* $|\text{succ}(a)| > 1$
4. (E) *evaporation* $|\text{succ}(a)| = 0$

We'll say that a cell is of type (x) if it has a function of type (x) ($(x) \in \{(C), (M), (D), (E)\}$).

Let $w = a_1 a_2 \dots a_n$ be a stage. If $w \Rightarrow w'$ we'll shall write $w' = \text{Succ}(w)$ understanding that $\text{Succ}(w) = \text{succ}(a_1) \text{succ}(a_2) \dots (\text{succ}(a_n))$.

a) if $\text{succ}(a_i) = \lambda \forall 1 \leq i \leq n$ then :

$w' = \text{succ}(a_1) \text{succ}(a_2) \dots \text{succ}(a_n) = \underbrace{\lambda \lambda \dots \lambda}_{n\text{-times}}$, and we'll write $\text{Succ}(w) = \lambda$.

b) if $i, 1 \leq i \leq n$ exists, such that $\text{succ}(a_i) = \lambda$ and $\text{succ}(a_j) \neq \lambda$, $i \neq j$ and $1 \leq j \leq n$, we'll write $w' = \text{Succ}(w) = \text{succ}(a_1) \dots \text{succ}(a_{i-1}) \text{succ}(a_{i+1}) \dots \text{succ}(a_n)$, so we're not going to write $\text{succ}(a_i)$.

c) if there exist more cells $i_1, \dots, i_k$ so that $\text{succ}(a_{i_j}) = \lambda, 1 \leq j \leq k$, we're not going to explicitly write these successors, but only those successors of the cells a_i so that $|\text{succ}(a_i)| \geq 1$.

Let $w_1 \Rightarrow w_2 \Rightarrow \dots \Rightarrow w_n$ be a growth of the $\mathcal{L}$ -system $L = (I, w_1, S)$. Let $f: N \rightarrow N$ be a function defined as $f(n) = |w_n|$. The function thus defined is called the function assigned to the $\mathcal{L}$ -system L for the given growth; and if the $\mathcal{L}$ -system is a deterministic one, the function thus defined will be simply called the function of the $\mathcal{L}$ -system.

For example, the function of the $\mathcal{L}$ -system from the second example if $f(n) = n$.

Proposition 1. If $L = (I, b, S)$ is deterministic, then there exists a function f assigned to the $\mathcal{L}$ -system.

Proof: by induction upon n

$f(1) = |b| = 1$ unique value (b -is unique)

$f(2) = |w_2| = n_2$ unique value ($w_2 = \text{succ}(b)$ -is unique)

Now, let's assume that $f(k) = |w_k| = n_k$ has a unique value and let's prove that $f(k+1)$ has a unique value too.

$w_k = a_1 a_2 \dots a_{n_k}$

$w_{k+1} = \text{succ}(a_1) \text{succ}(a_2) \dots \text{succ}(a_{n_k})$

As $\text{succ}(a_i)$ is unique for any $i, 1 \leq i \leq n_k$, it results that w_{k+1} is unique. Therefore :

$f(k+1) = |\text{succ}(a_1)| + |\text{succ}(a_2)| + \dots + |\text{succ}(a_{n_k})| = n_{k+1}$.

Observation 4. To the $\mathcal{L}$ -systems which are not deterministic we can't assign a function $f: N \rightarrow N$; but to any given growth of the respective $\mathcal{L}$ -systems we can assign a function as in the above mentioned case.

In what follows we'll use only deterministic $\mathcal{L}$ -systems.

II.3. The basic cell b is determined by taking into account the S system and the value of the function f_D for the first k consecutive values of its argument ($f_D(1), f_D(2), \dots, f_D(k)$), k depending on the S system, on the function f_D , on the type of basic cell and so on so forth.

We shall first exemplify the way $\mathcal{L}$ -systems are resolved.

Example 4. An $f_D(n) = n$ is given. Let's determine $L = (I, b, S)$ so that $f_L(n) = f_D(n)$. Let's determine I :

We assume that I is made of l cells $I = \{a_1, \dots, a_l\}$.

We assume we are at the n -th stage therefore $|w_n| = n, w = x_1 \dots x_n$ where $x_i \in I, 1 \leq i \leq n$. At the stage $n+1$ we'll have $n+1$ cells. Therefore, out of n cells of the n -th stage, $n-1$ will have to be of type (C) — see case 1, or of type (M) — see case 2, or of type (C) and (M) — see case 3; so $|\text{succ}(x_i)| = 1, \forall i = 1, 2, \dots, n-1$; and one of the cells — x_n to be of type (D) so that $|\text{succ}(x_n)| = 2$. Further, $y_1, \dots, y_n$ to be of type (C) or (M) and y_{n+1} be of type (D) and so on and so forth (see figure 3).

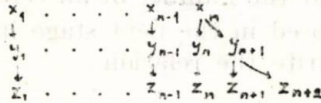


Fig. 3

Case 1. We need cells of the type (C) and let's denote them by a and cells of type (D) which will be denoted by b . Therefore $|I_1| = 2, I_1 = \{a, b\}$. The a cells are of the type (C) therefore $a \rightarrow a$. The b cells are of the type (D), according to the above written — see also figure 3, $b \rightarrow ab$. Therefore $|S| = 2, S = \begin{cases} a \rightarrow a & (C) \\ b \rightarrow ab & (D) \end{cases}$.

Because $|w_1| = 1, |w_2| = 2$ and $w_1 \Rightarrow w_2$ these results $w_1 = b$ (the basic cell is b because it must be of the type (D)).

Therefore: $L_1 = (\{a, b\}, b, S_1); S_1 = \begin{cases} a \rightarrow a & (C) \\ b \rightarrow ab & (D) \end{cases}; f_{L_1}(n) = n$.

We have obtained the $\mathcal{L}$ -system from the second example; a growth of the $\mathcal{L}$ -system is schematically represented in figure 1.

Case 2. We need cells of the type (M) which will be denoted by p , and the successor of p will be denoted by q ($|\text{succ}(p)| = 1$ and $p \neq \text{succ}(p)$); and, further on, q is also of the type (M) — and without restricting the generality to consider $\text{succ}(q) = p$. We still need cells of the type (D) which will be denoted by m ($|\text{succ}(m)| = 2$). Therefore: $|I_2| = 3; I_2 = \{m, p, q\}$. The p and q cells are of the type (M) $p \rightarrow q; q \rightarrow p$. The m cell is of the type (D) $m \rightarrow pm$. Therefore:

$$L_2 = (\{m, p, q\}, m, S_2); S_2 = \begin{cases} m \rightarrow pm & (D) \\ p \rightarrow q & (M) \\ q \rightarrow p & (M) \end{cases}; f_{L_2}(n) = n.$$

We have obtained the $\mathcal{L}$ -system from the third example; a growth of the $\mathcal{L}$ -system is schematically represented in figure 2.

Case 3. By combining the cases 1 and 2 we obtain :

$$L_3 = (\{m, p, q\}, m, S_3); S_3 = \begin{cases} m \rightarrow pm & (D) \\ p \rightarrow q & (M) \\ q \rightarrow q & (C) \end{cases}; f_{L_3}(n) = n.$$

We can see that the $\mathcal{L}$ -system L_1 is more efficient than the $\mathcal{L}$ -systems L_2 and L_3 .

Further, in order to resolve $\mathcal{L}$ -systems we'll proceed in the following way :

By $N_k(a_i)$ we'll denote the number of cells of the type a_i from the stage k .

Let's assume that at a particular stage, let's say k , we have cells of the type $a_1, a_2, \dots, a_l$. In this case $N_k(a_1) + N_k(a_2) + \dots + N_k(a_l) = n_k$, where n_k is the value of the k stage.

a) let's assume that the number of all cells of the type $a_{i_1}, \dots, a_{i_j}$ from the stage k are stored in the next stage in the number of cells of the type a_m . We can write the relation :

$$N_{k+1}(a_m) \leftarrow N_k(a_{i_1}) + \dots + N_k(a_{i_j}).$$

b) let's assume that we have several relations and $N(a_x)$ appears in one or several relations :

$$N_{k+1}(a_{m1}) \leftarrow N_k(a_{i1}) + \dots + N_k(a_x) + \dots + N_k(a_{ij1})$$

$$N_{k+1}(a_{m2}) \leftarrow N_k(a_{i2}) + \dots + N_k(a_x) + \dots + N_k(a_{ij2})$$

$$\dots \dots \dots$$

$$N_{k+1}(a_{mm}) \leftarrow N_k(a_{im}) + \dots + N_k(a_x) + \dots + N_k(a_{ijm})$$

where a_x may belong to the set $\{a_{m1}, a_{m2}, \dots, a_{mm}\}$.

The above mentioned relations point out that the number of cells of the type a_x from the stage k is to be found among the number of cells of the type a_{m1} from the stage $k+1$; is to be found among the number of cells of the type a_{m2} from the stage $k+1$; ...; is to be found among the number of the cells of the type a_{mm} from the stage $k+1$. Thus in the generative system S we'll have a succession of the form $a_x \rightarrow a_{m1} \dots a_{mm}$.

Example 5. Let $I = \{a, b, c\}$ and let's assume that we are at a stage of value k and we have the relations :

$$I = \begin{cases} N_{k+1}(a) \leftarrow N_k(b) + N_k(c) & (1) \\ N_{k+1}(b) \leftarrow N_k(a) & (2) \\ N_{k+1}(c) \leftarrow N_k(a) + N_k(b) & (3) \end{cases}$$

It can be noticed that (see figure 4)

$$\begin{aligned} N_k(a) = 4; N_k(b) = 3; N_k(c) = 3 & \quad \begin{cases} 6 = 3 + 3 & (1) \\ 4 = 4 & (2) \\ 7 = 4 + 3 & (3) \end{cases} \\ N_{k+1}(a) = 6; N_{k+1}(b) = 4; N_{k+1}(c) = 7 & \end{aligned}$$

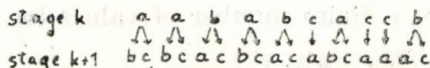


Fig. 4

Simplifying the notation we can write just a_x instead of $N_k(a_x)$ understanding by this notation that a_x represents the number of cells of type a_x , that is $N_k(a_x)$.

$$\text{The system I becomes: II} = \begin{cases} a \leftarrow b + c \\ b \leftarrow a \\ c \leftarrow a + b \end{cases}$$

Applying the considerations from the point b) we obtain:

$$S = \begin{cases} a \rightarrow bc \\ b \rightarrow \bar{a}c \\ c \rightarrow a \end{cases}$$

A growth is schematically represented in figure 4.

Observation 5. In order to solve the $\mathcal{L}$ -system which has a function $f = f_1 + f_2$, where f_1 and f_2 are functions whose $\mathcal{L}$ -systems are known: $L_1 = (I_1, b_1, S_1)$ and $L_2 = (I_2, b_2, S_2)$, respectively, we'll proceed in the following way:

a) if S_1 and S_2 are identical (in this case I_1 and I_2 are identical because S_1 and S_2 are identical and I_1 and I_2 are sets made of active cells). In this case $L = (I_1 \cup \{\sigma\}, \sigma, S_1 \cup \{\sigma \rightarrow b_1 b_2\})$, where σ is a fictitious stage $w_0 = \sigma$. Further, we'll have:

$$|w_1| = |b_1 b_2| = |b_1| + |b_2| = f_1(1) + f_2(1) = f(1)$$

$|w_2| = |\text{succ}(b_1)\text{succ}(b_2)| = f_1(2) + f_2(2) = f(2)$ and so on. See the example of III.3.3.

b) if S_1 and S_2 are not identical we must have $I_1 \cap I_2 = \emptyset$ (in case $I_1 \cap I_2 \neq \emptyset$ by simple operations (other notation) we can disjoint, the sets I_1 and I_2 , and in such a case we'll perform operations of the same kind on the generative system S , too). If $I_1 \cap I_2 = \emptyset$, $L = (I_1 \cup I_2 \cup \{\sigma\}, \sigma, S_1 \cup S_2 \cup \{\sigma \rightarrow b_1 b_2\})$ where σ is a fictitious stage $w_0 = \sigma$. See the example of III.3.4. The result can be generalized for $f = f_1 + f_2 + \dots + f_r$, $r \geq 2$.

Further on, we'll present some general methods for the resolution of some types of $\mathcal{L}$ -systems, obtaining without fail some solution (an

$\mathcal{L}$ -system) through these method, which in particular examples of f_D functions is possible not to be the most efficient $\mathcal{L}$ -system (see example of III.3.4).

III. Applications

III.1. $f(n)$ takes a finite number of values for $n \in N$.

III.1.1. $f(n) = k$.

In this case we need a fictitious stage w_0 , $|w_0| = 1$, let's say $w_0 = \sigma$. Because $|w_1| = k$ we have the succession $\sigma \rightarrow \underbrace{aa \dots a}_{k\text{-times}}$.

Therefore :

$$L = (\{a, \sigma\}, \sigma, S); S = \begin{cases} \sigma \rightarrow \underbrace{aa \dots a}_{k\text{-times}} \\ a \rightarrow a \end{cases}$$

III.1.2. $f(1) = n_1; f(2) = n_2; \dots; f(k) = n_k; f(l) = 0 \quad \forall l, k < l, l \in N$

In this case we need a fictitious stage $w_0 = \sigma$ and productions of the form : $\sigma \rightarrow a_1^1 \dots a_{n_1-1}^1 s_1; s_j \rightarrow a_1^{j+1} \dots a_{n_{j+1}-1}^{j+1} s_{j+1}, 1 \leq j \leq k-1, s_k \rightarrow \lambda, a_j^m \rightarrow \lambda, \forall j, m$.

$$\begin{aligned} \text{Therefore : } \text{succ}(\sigma) &= a_1^1 \dots a_{n_1-1}^1 s_1 & |\text{succ}(\sigma)| &= n_1 \\ \text{succ}(s_j) &= a_1^{j+1} \dots a_{n_{j+1}-1}^{j+1} s_{j+1} & |\text{succ}(s_{j+1})| &= n_{j+1} \quad 1 \leq j \leq k-1 \\ \text{succ}(s_k) &= \lambda & |\text{succ}(s_k)| &= 0 \end{aligned}$$

Therefore : $L = (I \cup \{\sigma\}, \sigma, S)$

where : $I = \{a_1^1, \dots, a_{n_1-1}^1, \dots, a_1^k, \dots, a_{n_k-1}^k, s_1, \dots, s_k\}$ and

$$S = \begin{cases} \sigma \rightarrow a_1^1 \dots a_{n_1-1}^1 s_1 \\ s_j \rightarrow a_1^{j+1} \dots a_{n_{j+1}-1}^{j+1} s_{j+1} & 1 \leq j \leq k-1 \\ s_k \rightarrow \lambda \\ a_j^m \rightarrow \lambda \quad \forall j, m \end{cases}$$

III.1.3. $f(n+k) = f(n) \quad \forall n \in N$

Therefore : $f(1) = n_1, f(2) = n_2, \dots, f(k) = n_k, f(k+1) = f(1) = n_1, f(2k) = n_k, \dots$ The idea is the same as in the previous case with the only difference that $s_k \rightarrow a_1^1 \dots a_{n_1-1}^1 s_1$ (therefore, we can consider that s_k is the basic cell).

Therefore : $L = (I, s_k, S)$ where :

$$I = \{a_1^1, \dots, a_{n_1-1}^1, \dots, a_1^k, \dots, a_{n_k-1}^k, s_1, \dots, s_k\} \text{ and}$$

$$S = \begin{cases} s_k \rightarrow a_1^1 \dots a_{n_1-1}^1 s_1 \\ s_j \rightarrow a_1^{j+1} \dots a_{n_{j+1}-1}^{j+1} s_{j+1} & 1 \leq j \leq k-1 \\ a_j^m \rightarrow \lambda \quad \forall j, m \end{cases}$$

III.2.1. $f(n) = k^n \quad k > 1$

Let's assume that we are at the n -th stage where there are k^n cells, at the next stage there are $k^{n+1} = k \times k^n$ cells. Therefore, each cell must divide, at the next stage, into k cells. As for the initial stage we need k cells, we can consider a fictitious stage $w_0 = \sigma, w_0 \Rightarrow w_1, |w_1| = k$. Therefore we obtain the $\mathcal{L}$ -system:

$L = (\{\sigma, a\}, \sigma, S)$ where

$$S = \begin{cases} \sigma \rightarrow a \dots a \\ \quad k\text{-times} \\ a \rightarrow a \dots a \\ \quad k\text{-times} \end{cases}$$

III.2.2. $f(n) = mk^n$ $m > 1, k > 1$ (the observation 5 is taken into account).

In this case: $|w_1| = f(1) = mk$. We'll consider a fictitious stage $w_0 = \sigma, |w_0| = |\sigma| = 1$ and $\sigma \rightarrow aa \dots aa$ and $a \rightarrow aa \dots a$.
 $m \times k\text{-times} \quad k\text{-times}$

Therefore: $L = (\{\sigma, a\}, \sigma, S)$ where:

$$S = \begin{cases} \sigma \rightarrow aa \dots a \\ \quad m \times k\text{-times} \\ a \rightarrow aa \dots a \\ \quad k\text{-times} \end{cases}$$

$$\text{III.3. } f(n) = \sum_{i=0}^r k_i n^i \quad k_i \in \mathbb{Z}, k_r > 0$$

III.3.1. $f(n) = k$ (see III.1.1.)

$$L = (\{a, \sigma\}, \sigma, S)$$

$$S = \begin{cases} \sigma \rightarrow aa \dots aa \\ \quad k\text{-times} \\ a \rightarrow a \end{cases}$$

III.3.2. $f(n) = n$ (see the example 2 and the example 4)

$$L = (\{a, b\}, b, S)$$

$$S = \begin{cases} a \rightarrow a \\ b \rightarrow ab \end{cases}$$

III.3.3. $f(n) = k \times n$ (the observation 5 is taken into account)

In this case $|w_1| = k$. We'll consider a fictitious stage $w_0, |w_0| = 1$ therefore $w = \sigma, \sigma \rightarrow s_1 \dots s_1$, where s_1 is the basic cell for the $\mathcal{L}$ -system L_1 , so that:

$$f_{L_1}(n) = n \quad (f(n) = f_{L_1}(n) + \dots + f_{L_1}(n))$$

$$k\text{-times}$$

Therefore: $L = (\{\sigma, a, b\}, \sigma, S)$ where

$$S = \begin{cases} \sigma \rightarrow bb \dots b \\ \quad k\text{-times} \\ a \rightarrow a \\ b \rightarrow ab \end{cases}$$

III.3.4. $f(n) = n^2$

Let's assume that we are at the $(n+1)$ -th stage, $|w_{n+1}| = (n+1)^2$ and $w_{n+1} \Rightarrow w_{n+2}$

$$|w_{n+1}| = (n+1)^2 = n^2 + 2n + 1$$

$$|w_{n+2}| = (n+1+1)^2 = \underset{a}{(n+1)^2} + 2\underset{b}{(n+1)} + \underset{c}{1}$$

We need, therefore, cells of three types:

- type a to model the growth of type n^2
- type b to model the growth of type $k \times n$ ($2n$)
- type c to model the growth of constant type ($k = 1$)

At the $(n+2)$ -th stage all cells from the previous stage (the $(n+1)$ -th stage, respectively) pass into cells of type a . We can, therefore, write $a \leftarrow a + b + c$.

At the $(n+2)$ -th growth stage the cells of the type b represent the growth of the type $2n$, therefore we could write $b \leftarrow b$; but taking into account III.3.2. we can notice that b stand for cells of the type b_1 and b_2 which are necessary for the growth of the type $2n$. Therefore: $\begin{cases} b_1 \leftarrow b_1 \\ b_2 \leftarrow b_1 + b_2 \end{cases}$

Similarly, $c \leftarrow c$.

Therefore:

$$I = \begin{cases} a \leftarrow a + b_1 + b_2 + c \\ b_1 \leftarrow b_1 \\ b_2 \leftarrow b_1 + b_2 \\ c \leftarrow c \end{cases} \Rightarrow S_1 = \begin{cases} a \rightarrow a \\ b_1 \rightarrow ab_1b_2 \\ b_2 \rightarrow ab_2 \\ c \rightarrow ac \end{cases}$$

Because $|w_1| = 1$ and $|w_2| = 4 = 1 + 2 \times 1 + 1$ we need the succession $s \rightarrow ab_1b_1c$. Therefore $L = (\{a, b_1, b_2, c, s\}, s, S)$ where:

$$S = \begin{cases} s \rightarrow ab_1b_1c \\ a \rightarrow a \\ b_1 \rightarrow ab_1b_2 \\ b_2 \rightarrow ab_2 \\ c \rightarrow ac \end{cases}$$

Observation 6. A more refined solution to obtain a more efficient $\mathcal{L}$ -system is obtained by taking into account that: $n^2 + 2(n+1) + 1 = n^2 + 2n + 2 + 1$. Therefore:

$$I = \begin{cases} a \leftarrow a + b + c \\ b \leftarrow b + 2c \\ c \leftarrow c \end{cases} \Rightarrow S = \begin{cases} a \rightarrow a \\ b \rightarrow ab \\ c \rightarrow abbc \end{cases}$$

As $|w_1| = 1$, $|w_2| = 4$ there results that c is the basic cell.

Therefore : $L = (\{a, b, c\}, c, S)$ where

$$S = \begin{cases} a \rightarrow a \\ b \rightarrow ab \\ c \rightarrow abbc \end{cases}$$

III.3.5. $f(n) = kn^2$

We take into account the observation 5 and the method is similar to II.3.3. In this case $|w_1| = k$. We'll consider a fictitious stage $w_0 = \sigma$, $\sigma \rightarrow \underset{k\text{-times}}{ss \dots ss}$ where s is a basic cell for the $\mathcal{L}$ -system L_2 so that $f_{L_2}(n) = n^2$.

Therefore $L = (\{\sigma, a, b, c\}, \sigma, S)$ where

$$S = \begin{cases} \sigma \rightarrow \underset{k\text{-times}}{ss \dots ss} \\ s \rightarrow ab_1b_1c \\ a \rightarrow a \\ b_1 \rightarrow ab_1b_2 \\ b_2 \rightarrow ab_2 \\ c \rightarrow ac \end{cases}$$

III.3.6. $f(n) = n^3$

The method is similar to the case III.3.4.

Let's assume that we are at the $(n+1)$ -th stage that is

$$|w_{n+1}| = (n+1)^3, w_{n+1} \Rightarrow w_{n+2}.$$

$$|w_{n+1}| = (n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$|w_{n+2}| = (n+1+1)^3 = (n+1)^3 + 3(n+1)^2 + 3(n+1) + 1$$

Therefore we need four types of cells :

- type m to model the growth of the type n^3
- type p to model the growth of the type n^2 ($3n^2$)
- type q to model the growth of the type n ($3n$)
- type r to model the growth of constant type (1)

Therefore, we can write the relations :

$$I = \begin{cases} m \leftarrow m + p + q + r \\ p \leftarrow p \\ q \leftarrow q \\ r \leftarrow r \end{cases} \Rightarrow S_1 = \begin{cases} m \rightarrow m \\ p \rightarrow mp \\ q \rightarrow mq \\ r \rightarrow mr \end{cases}$$

But p represents (models) a growth of the type n^2

$$p : \begin{cases} s \rightarrow ab_1b_1c \\ a \rightarrow a \\ b_1 \rightarrow ab_1b_2 \\ b_2 \rightarrow ab_2 \\ c \rightarrow ac \end{cases} \Rightarrow mp : \begin{cases} s \rightarrow mab_1b_1c \\ a \rightarrow ma \\ b_1 \rightarrow mab_1b_2 \\ b_2 \rightarrow mab_2 \\ c \rightarrow mac \end{cases}$$

Similarly, q represents a growth of the type n

$$q: \begin{cases} x \rightarrow xy \\ y \rightarrow y \end{cases} \Rightarrow mq: \begin{cases} x \rightarrow mxy \\ y \rightarrow my \end{cases}$$

We consider t as a basic cell $|w_1| = |t| = 1, w_1 \Rightarrow w_2, w_2 = 8 = 1 + 3 + 3 + 1$
Therefore $L = (\{t, m, s, a, b_1, b_2, c, x, y, r\}, t, S)$ where:

$$S = \begin{cases} t \rightarrow mssssxxxr \\ m \rightarrow m \\ s \rightarrow mab_1b_1c \\ a \rightarrow ma \\ b_1 \rightarrow mab_1b_2 \\ b_2 \rightarrow mab_2 \\ c \rightarrow mac \\ x \rightarrow mxy \\ y \rightarrow my \\ r \rightarrow mr \end{cases} \begin{cases} m \rightarrow m \\ p \rightarrow mp \\ q \rightarrow mq \\ r \rightarrow mr \end{cases}$$

A growth of the $\mathcal{L}$ -system L is schematically represented in fig. 5.

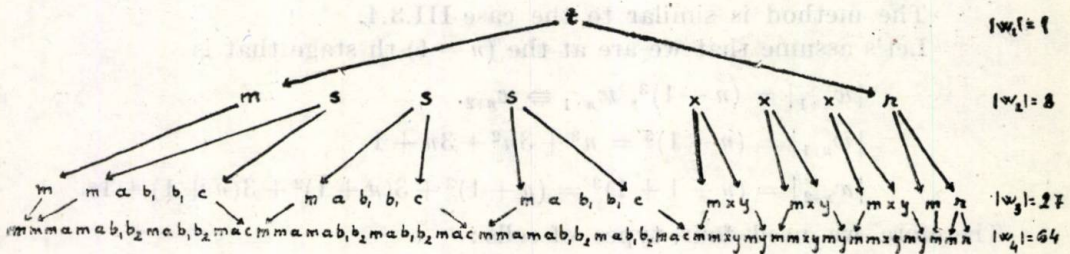


Fig. 5

It can be noticed that:

1. $N_k(m) = (k-1)^3$ See figure 5, where in the third stage, for example, we have $(3-1)^3 = 8$ cells of the type m , in the second stage $(2-1)^3 = 1$ cell of type m .
2. $N_k(s) + N_k(a) + N_k(b_1) + N_k(b_2) + N_k(c) = 3(k-1)^2$
See same figure where in the third stage we have:

$$N_3(s) = 0$$

$$N_3(a) = 3$$

$$N_3(b_1) = 6$$

$$N_3(b_2) = 0$$

$$N_3(c) = 3$$

$$12 = 3 \times (3-1)^2$$

3. $N_k(x) + N_k(y) = 3(k-1)$

See same figure where in the third stage we have :

$$\begin{array}{r} N_3(x) = 3 \\ N_3(y) = 3 \\ \hline 6 = 3 \times (3 - 1) \end{array}$$

4. $N_k(r) = 1$ See same figure.

III.3.7. $f(n) = kn^3$

We take into account observation 5 and the method is similar to III.3.3. and III.3.5.

In this case $|w_1| = k$. We'll consider a fictitious stage $w_0 = \sigma$, $\sigma \rightarrow \underset{k\text{-times}}{tt \dots tt}$ where t is the basic cell for the $\mathcal{L}$ -system L_3 so that $f_{L_3}(n) = n^3$.

Therefore : $L = (\{\sigma, t, m, s, a, b_1, b_2, c, x, y, r\}, \sigma, S)$

$$S = \begin{cases} \sigma \rightarrow \underset{k\text{-times}}{tt \dots t} \\ t \rightarrow mssxxr \\ m \rightarrow m \\ s \rightarrow mab_1b_1c \\ a \rightarrow ma \\ b_1 \rightarrow mab_1b_2 \\ b_2 \rightarrow mab_2 \\ c \rightarrow mac \\ x \rightarrow mxy \\ y \rightarrow my \\ r \rightarrow mr \end{cases}$$

By applying the method in the previously way, we can find $\mathcal{L}$ -systems so that the function $f_i(n) = k_i n^i$ can be modelled for any $i \in N$. It can be noticed that, in order to solve the $\mathcal{L}$ -system which has the assigned function $f_L(n) = k_i n^i$, first we'll have to successively solve the $\mathcal{L}$ -systems for the assigned function $f_j(n) = kn^j \forall j < i$

III.3.8. $f(n) = \sum_{i=0}^r k_i n^i, k_r > 0$

A. Let's consider the case when all the coefficients k_i are strictly positive, $k_i \in N^*$.

In this case one must first determine the $\mathcal{L}$ -system for the functions $f_i(n) = k_i n^i, 1 \leq i \leq r$, and next, taking into consideration observation 5, we determine the $\mathcal{L}$ -system L so that $f_L = f$.

Example 6.

$$f(n) = 2n^2 + 3n + 1$$

From III.3.5. there results

$$L_1 = (\{\sigma, a, b, c\}, \sigma, S_1)$$

$$S_1 = \begin{cases} \sigma \rightarrow cc \\ a \rightarrow a \\ b \rightarrow ab \\ c \rightarrow abbc \end{cases}$$

From III.3.3. there results

$$L_2 = (\{\tau, x, y\}, \tau, S_2)$$

$$S_2 = \begin{cases} \tau \rightarrow xxx \\ x \rightarrow xy \\ y \rightarrow y \end{cases}$$

From III.3.1 there results

$$L_3 = (\{\xi, m\}, \xi, S_3)$$

$$S_3 = \begin{cases} \xi \rightarrow m \\ m \rightarrow m \end{cases}$$

In this case we need two fictitious stages w_{00} and w_{01} ,

$$w_{00} = \eta, |w_{00}| = |\eta| = 1, \eta \rightarrow \sigma\tau\xi, w_{01} = \sigma\tau\xi, |w_{01}| = 3.$$

Therefore :

$$L = (\{\eta, \sigma, \tau, \xi, a, b, c, x, y, m\}, \eta, S) \text{ where } S = S \cup {}_1S_2 \cup S_3 \cup \{\eta \rightarrow \sigma\tau\xi\}$$

A growth of the $\mathcal{L}$ -system L is schematically represented in fig. 6.

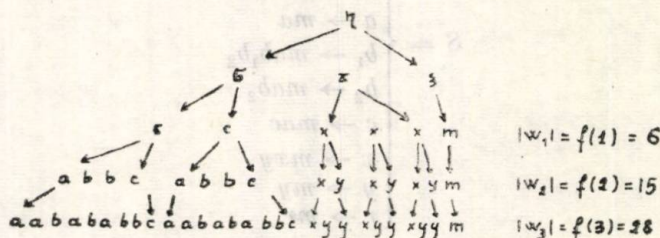


Fig. 6

In fig. 6 we notice that :

$$N_k(a) + N_k(b) + N_k(c) = 2k^2$$

$$N_k(x) + N_k(y) = 3k$$

$$N_k(m) = 1$$

B. Let's consider the case when $k_i \in \mathbb{Z}$, $0 \leq i \leq r$.

In this case k_r = the coefficient of the member of the maximum degree must be strictly positive, therefore, there must be a first threshold $l \in \mathbb{N}$ so that $f(n) > 0 \quad \forall n > l$. (this threshold exists because $\lim_{n \rightarrow +\infty} f(n) = +\infty$).

As a rule, we'll consider the growth function f , for bigger values than l , and for values between 1 and l , it will be modelled according to requirements from one case to another.

Lemma 1 Let $p(n) = \sum_{i=1}^r k_i n^i$, $k_i \in \mathbb{Z}$, $0 \leq i \leq r$, $k_r > 0$.

Let $0 \leq j < r$ be so that $k_j \in \mathbb{Z}$, $k_{j+1} > 0, \dots, k_r > 0$.

Then there is $p_1(n) = \sum_{i=0}^r h_i n^i$, $h_i \in Z$, $0 \leq i \leq r$ with the property:

$$h_r = k_r > 0, h_{r-1} > k_{r-1} > 0, \dots, h_{j+1} > k_{j+1} > 0, h_j > k_j \text{ and } p_1(n) = p(n+1)$$

$$\text{Proof: } p(n) = \sum_{i=0}^r k_i n^i = \sum_{i=j+1}^r k_i n^i + \sum_{i=0}^{j-1} k_i n^i + k_j n^j$$

$$\begin{aligned} p(n+1) &= \sum_{i=j+1}^r k_i (n+1)^i + k_j (n+1)^j + \sum_{i=0}^{j-1} k_i (n+1)^i = \\ &= \sum_{i=j+1}^r k_i \left(\sum_{s=0}^i \binom{i}{s} n^{i-s} \right) + k_j \left(\sum_{s=0}^j \binom{j}{s} n^{j-s} \right) + \sum_{i=0}^{j-1} k_i \left(\sum_{s=0}^i \binom{i}{s} n^{i-s} \right) = \\ &= k_r \binom{0}{r} n^r + \left(k_r \binom{1}{r} + k_{r-1} \binom{0}{r-1} \right) n^{r-1} + \left(k_r \binom{2}{r} + k_{r-1} \binom{1}{r-1} + \right. \\ &\quad \left. + k_{r-2} \binom{0}{r-2} \right) n^{r-2} + \dots + \left(k_r \binom{r-j}{r} + \dots + k_{j+1} \binom{1}{j+1} + \right. \\ &\quad \left. + k_j \binom{0}{j} \right) n^j + \dots + \left(k_r \binom{r-1}{r} + \dots + k_1 \binom{0}{1} \right) n + k_r \dots + k_0 \end{aligned}$$

Let's denote:

$$h_j = \sum_{i=j}^r k_i \binom{i-j}{i}$$

In this case:

$$h_r = k_r > 0$$

$$h_{r-1} = \sum_{i=r-1}^r k_i \binom{i-r+1}{i} = k_{r-1} + k_r \binom{1}{r} > k_{r-1} > 0$$

$$h_{r-2} = \sum_{i=r-2}^r k_i \binom{i-r+2}{i} = k_{r-2} + \sum_{i=r-1}^r k_i \binom{i-r+2}{i} > k_{r-2} > 0$$

$$h_{j+1} = \sum_{i=j+1}^r k_i \binom{i-j-1}{i} = k_{j+1} + \sum_{i=j+2}^r k_i \binom{i-j-1}{i} > k_{j+1} > 0$$

$$h_j = \sum_{i=j}^r k_i \binom{i-j}{i} = k_j + \sum_{i=j+1}^r k_i \binom{i-j}{i} > k_j$$

Therefore $p_1(n) = \sum_{i=0}^r h_i n^i$. Obviously $p_1(n) = p(n+1)$

Example 7

$$\begin{aligned} p(n) &= n^3 - 5n^2 + 5n + 4 & j &= 2, k_2 < 0, k_3 > 0 \\ p_1(n) &= p(n+1) = n^3 - 2n^2 - 2n + 5 & h_3 &= k_3 = 1 \\ & & h_2 &= -2 > -5 = k_2 \end{aligned}$$

Lemma 2: Let $p(n) = \sum_{i=0}^r k_i n^i$, $k_i \in \mathbb{Z}$, $0 \leq i \leq r$, $k_r > 0$.

Let $0 < j < r$ be so that $k_j < 0$, $k_{j+1} > 0$, ..., $k_r > 0$.

Then, there is $l \in \mathbb{N}$, $l > 0$ and $p_l(n) = \sum_{i=0}^r h_i n^i$, $h_i \in \mathbb{Z}$, $0 \leq i \leq r$, with the property:

$$h_r = k_r > 0; h_{r-1} > k_{r-1} > 0; \dots; h_{j+1} > k_{j+1} > 0; h_j > 0 > k_j \text{ and } p_l(n) = p(n+l)$$

Proof: Lemma 1 is applied for a finite number of times (that number will be l), obtaining thus the coefficient of the member of j degree bigger and bigger until it becomes positive.

Example 8: $p(n) = n^3 - 5n^2 + 5n + 4$

$$L1: l = 1 \quad p_1(n) = p(n+1) = n^3 - 2n^2 - 2n + 5 \quad h_2 = -2 < 0$$

$$L1: l = 2 \quad p_2(n) = p_1(n+1) = n^3 + n^2 - 3n + 2 \quad h_2 = 2 > 0$$

$$\text{Therefore } l=2 \quad p_2(n) = n^3 + n^2 - 3n + 2 \quad h_3 = k_r = 1 > 0$$

$$\text{Obviously} \quad p_2(n) = p(n+2) \quad h_2 = 2 > 0 > -5 = k_2$$

Theorem 1: Let $p(n) = \sum_{i=0}^r k_i n^i$, $k_i \in \mathbb{Z}$, $0 \leq i \leq r$, $k_r > 0$.

Then there is $l \in \mathbb{N}$, $l \geq 0$ and $p_l(n) = \sum_{i=0}^r h_i n^i$, $h_i \in \mathbb{N}$, $0 \leq i \leq r$ $\left. \vphantom{\sum_{i=0}^r h_i n^i} \right\} (*)$
and $p_l(n) = p(n+l)$

Proof:

If $k_i > 0$, $0 \leq i \leq r$, then $h_i = k_i$, $0 \leq i \leq r$, $l = 0$ and $p_0(n) = p(n)$.

If $\exists k_i < 0$, $0 \leq i \leq r-1$, then we are applied lemma 1 and lemma 2 several times (l -times) until we get the conditions (*).

Example 9: $p(n) = n^3 - 5n^2 + 5n + 4$; $j = 2$, $k_2 < 0$, $k_3 > 0$.

Here's applied lemma 2 (see example 8) and we obtain:

$$l = 2 \quad p_2(n) = n^3 + n^2 - 3n + 2 \quad j = 1, k_1 < 0, k_2 > 0, k_3 > 0$$

$$L2: l = 3 \quad p_3(n) = n^3 + 4n^2 + 2n + 1$$

$$k_3 > 0, k_2 > 0, k_1 > 0, k_0 > 0$$

$$\text{Therefore } l = 3 \quad p_3(n) = n^3 + 4n^2 + 2n + 1 = p(n+3)$$

In conclusion, if a growth function $f(n) = \sum_{i=1}^r k_i n^i$, $k_i \in \mathbb{Z}$, $0 \leq i \leq r$,

$k_r > 0$ is given and an $\mathcal{L}$ -system $L = (I, b, S)$ is to be obtained so that $f_L = f$ we'll proceed in the following way:

We calculate l and $f_l(n) = \sum_{i=0}^r h_i n^i$, $h_i \in \mathbb{Z}$, $h_i > 0$, $0 \leq i \leq r$, and for this growth function f_l the $\mathcal{L}$ -system will be found as in the A case. We must take into account that $f_l(n) = f(n+l)$.

In such a case if we are interested in modelling the first l stages $w_1, \dots, w_l$ (the stage $l+1$, therefore w_{l+1} , being the „first stage” for the f_l function); their values can be given explicitly or not. (If $[w_1], \dots, [w_l]$ are not given then $[w_i] = f(i)$, $1 \leq i \leq l$ and $f(i) > 0$, $1 \leq i \leq l$ in order to have a growth).

a) $l=0$ we are in the A case.

b) $l=1$. If $|w_1| < r$ then we can't model stage 1; we need two fictitious stages w_{00}, w_{01} (the same as in the A case, see example 6) and in this case $f_l(1) = f(2)$, therefore, the growth from the second stage is modelled.

If $|w_1| \geq r$ we can consider the fictitious stage w_{01} ($|w_{01}| = r \leq |w_1|$) to be stage w_1 , to which the cells of type (E) may be eventually added. We still need one more fictitious stage $w_{00} = \eta$, $|w_{00}| = 1$.

Example 10: $f(n) = 2n^2 - 3n + 2$ $|w_1| = 6$ is initial given.

$$l = 1, f_1(n) = f(n+1) = 2n^2 + n + 1$$

Similarly, as in example, 6, we have:

$$S_1 = \begin{cases} \sigma \rightarrow cc \\ a \rightarrow a \\ b \rightarrow ab \\ c \rightarrow abbc \end{cases} \quad S_2 = \begin{cases} \tau \rightarrow x \\ x \rightarrow xy \\ y \rightarrow y \end{cases} \quad S_3 = \begin{cases} \xi \rightarrow m \\ m \rightarrow m \end{cases}$$

According to the previous observations we have: $u_0 = \eta$,

$\eta \rightarrow \sigma\tau\xi fff$, $f \rightarrow \lambda$. Therefore $L = (\{\eta, \sigma, \tau, \xi, a, b, c, x, y, m, f\}, \eta, S)$ where $S = S_1 \cup S_2 \cup S_3 \cup \{\eta \rightarrow \sigma\tau\xi fff\} \cup \{f \rightarrow \lambda\}$.

A growth of the $\mathcal{L}$ -system L is schematically represented in fig. 7.

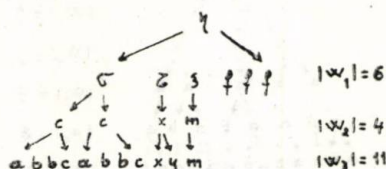


Fig. 7

c) $l > 1$ If $|w_l| < r$ we need two fictitious stages w_{00}, w_{01} (as in the A case, see example 6), in this case $f_l(1) = f(l+1)$, therefore the growth starts from the $(l+1)$ -th stage.

If $|w_l| \geq r$ we have a similar approach as in II.1.2. In this case $w_1, \dots, w_{l-1}$ are given; next, we proceed in the same manner as in the previous example for the growth sequence $w_{l-1} \Rightarrow u_l \Rightarrow u_{l+1}$ and further this growth will be given by the function $f_l(n)$.

$$\text{III.4. } f(n) = \sum_{i=1}^r k_i f(n-i)$$

In this case $f(1) = n_1, f(2) = n_2, \dots, f(r) = n_r$ are initially given.

$$\text{III.4.1. } r = 1 \quad f(n) = kf(n-1) \quad f(1) = m$$

We can notice that $f(n) = kf(n-1) = k^2f(n-2) = \dots = k^n f(1) = mk^n$. Therefore $f(n) = mk^n$, but this case was considered in III.2.2 ($k \neq 1$) or III.1.1 ($k = 1$).

III.4.2. $r = 2$ $f(n) = k_1 f(n-1) + k_2 f(n-2)$ $f(1) = m_1$; $f(2) = m_2$
 a) $k_1 = k_2 = 1$, $m_1 = m_2 = 1$. Therefore: $f(n) = f(n-1) + f(n-2)$
 $(1) = f(2) = 1$ Let's assume that the growth is in the s -th stage and $I = \{a, b, c, \dots, t\}$. According to our previous notations we obtain:

$$N_s(a) + N_s(b) + \dots + N_s(t) = f(s), \text{ therefore, } a + b + \dots + t = f(s).$$

Let's consider the growth sequence: $w_s \Rightarrow w_{s+1} \Rightarrow w_{s+2}$; $f(s+2) = f(s+1) + f(s)$. Let us assume that $a \leftarrow a + b + \dots + t$; $b \leftarrow a$. Therefore in the $(s+1)$ -th stage they will be $f(s)$ cells of the type a because $a \leftarrow a + b + \dots + t$; in the $(s+2)$ -th stage they will be $f(s+1)$ cells of type a and $f(s)$ cells of type b because $b \leftarrow a$. Therefore $I = \{a, b\}$. Using the next scheme we may represent this growth:

$$\begin{aligned} N_s(a) + N_s(b) &= f(s) \\ N_{s+1}(b) + N_{s+1}(a) &= f(s+1) \\ N_{s+2}(a) + N_{s+2}(b) &= f(s+2) \\ I = \begin{cases} a \leftarrow a + b \\ b \leftarrow a \end{cases} &\Rightarrow S = \begin{cases} a \rightarrow ab \\ b \rightarrow a \end{cases} \end{aligned}$$

As $m_1 = m_2 = 1 \Rightarrow w_1 = b \Rightarrow a = u_2$

Therefore: $L = (\{a, b\}, b, S)$

A growth of the $\mathcal{L}$ -system L is given in the figure 8.

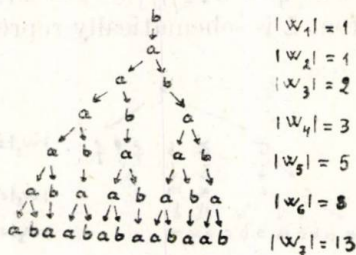


Fig. 8

We can notice that:

$$N_k(a) = f(k-1) = |w_{k-1}| \quad \forall k \in \mathbb{N}$$

$$N_k(b) = f(k-2) = |w_{k-2}| \quad \forall k \in \mathbb{N}$$

$$N_k(a) + N_k(b) = f(k) = f(k-1) + f(k-2) \quad \forall k \in \mathbb{N}$$

b) $k_1 = 1$, $k_2 = 1$; $|w_1| = m_1 \geq 1$, $|w_2| = m_2 \geq 1$

Therefore $f(n) = f(n-1) + f(n-2)$ and $f(1) = m_1$, $f(2) = m_2$.

We'll take into account III.1.2 in order to model the growth sequence $w_1 \Rightarrow w_2$. We take into account the following conditions:

1. in the third stage: $|w_3| = m_1 + m_2$, $N_3(a) = m_2$, $N_3(b) = m_1$

c1) $k_1 = k_2$, then $k_2b \leftarrow k_1a$ is equivalent with $b \leftarrow a$.

$$I = \begin{cases} a \leftarrow k_1a + k_1b \\ b \leftarrow a \end{cases} \Rightarrow S = \begin{cases} a \rightarrow a \dots a b \\ \quad \quad \quad k_1\text{-times} \\ b \rightarrow aa \dots a \\ \quad \quad \quad k_1\text{-times} \end{cases}$$

See example 10 where $k_1 = k_2 = 1$.

c2) $k_1 < k_2$, $k_2b \leftarrow k_1a$

This growth sequence is schematically represented in figure 10. It can be noticed that in this case the cells of the type a are folled into

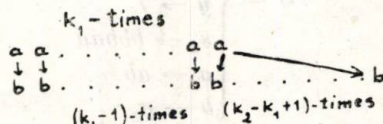


Fig. 10

two classes. Therefore $I = \{a_1, a_2, b\}$ and $b \leftarrow a_1$; $bb \dots bb \leftarrow a_2$.
(k_2-k_1+1)-times

Therefore :

$$I = \begin{cases} a \leftarrow k_1a + k_2b \\ k_2b \leftarrow k_1a \end{cases} \quad a \leftarrow k_1a \Rightarrow a \leftarrow aa \dots aa, \text{ but } k_1a \leftarrow k_2b, \text{ that is, one} \\ \quad \quad \quad k_1\text{-times} \\ \text{cell from } k_1 \text{ of the type } a \text{ is of the type } a_2 \text{ and the other cells are of} \\ \text{the type } a_1; \text{ similarly, } a \leftarrow b \Rightarrow b \rightarrow a_1 \dots a_1a_2 \\ \quad \quad \quad (k_1-1)\text{-times}$$

It results :

$$II = \begin{cases} b \leftarrow a_1 \\ bb \dots bb \leftarrow a_2 \\ \quad \quad \quad (k_2-k_1+1)\text{-times} \end{cases} \left\{ \begin{array}{l} k_2b \leftarrow k_1a \\ a \leftarrow k_1a \\ a \leftarrow k_2b \end{array} \right.$$

Therefore :

$$S = \begin{cases} a_1 \rightarrow a_1 \dots a_1a_2b \\ \quad \quad \quad (k_1-1)\text{-times} \\ a_2 \rightarrow a_1 \dots a_1a_2bb \dots bbbbbb \\ \quad \quad \quad (k_1-1)\text{-times} \quad (k_2-k_1+1)\text{-times} \\ b \rightarrow a_1 \dots a_1a_2 \\ \quad \quad \quad (k_2-1)\text{-times} \end{cases}$$

Example 11: $f(n) = 2f(n-1) + 3f(n-2)$ $f(1) = 2$, $f(2) = 1$

For $k = 3$ we get $\left. \begin{array}{l} N_3(a) = k_1f(2) = 2 \\ N_3(b) = k_2f(1) = 6 \end{array} \right\} f(3) = 8$

For $k = 4$ we get $\left. \begin{array}{l} N_4(a) = 16 \\ N_4(b) = 3 \end{array} \right\} f(4) = 19$

In figure 11, we can see that in order to have $N_4(a) = 16$, $N_4(b) = 3$, there will have to be a cell of type a_1 and a cell of type a_2 in the third stage.

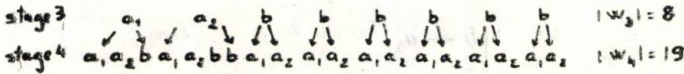


Fig. 11

According to III.1.2 we need a fictitious stage $w_0 = \eta$. Therefore $L = (\{\eta, s, y, t, a_1, a_2, b\}, \eta, S)$ where :

$$S = \begin{cases} \eta \rightarrow sy \\ y \rightarrow \lambda \\ s \rightarrow t \\ t \rightarrow a_1 a_1 b b b b b b \\ a_1 \rightarrow a_1 a_2 b \\ a_2 \rightarrow a_1 a_2 b b \\ b \rightarrow a_1 a_2 \end{cases}$$

A growth of the $\mathcal{L}$ -system L is given in figure 12.

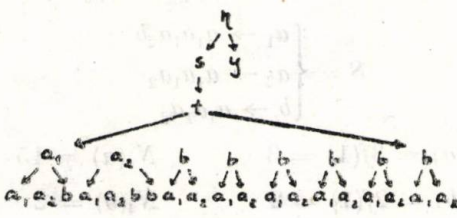


Fig. 12

c3) $k_1 > k_2, k_2 b \leftarrow k_1 a$

This growth sequence is schematically represented in figure 13. In this

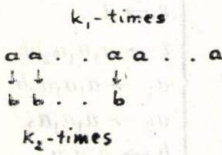


Fig. 13

case the cells of the type a are divided into cells of the type a_1 and cells of the type a_2 , therefore $I = \{a_1, a_2, b\}$.

$$b \leftarrow a_1 \text{ and } \lambda \leftarrow a_2$$

Therefore :

$$I = \begin{cases} a \leftarrow k_1 a + k_2 b \\ k_2 b \leftarrow k_1 a \end{cases}$$

$a \leftarrow k_1 a \Rightarrow a \rightarrow \underbrace{aaa \dots a}_{k_1 \text{-times}}$, but $k_1 a \rightarrow k_2 b \Rightarrow a \rightarrow \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}}$
 Similarly $b \rightarrow \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}}$

It results :

$$\text{II} = \begin{cases} b \leftarrow a_1 \\ \lambda \leftarrow a_2 \\ \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}} \leftarrow a_1 \\ \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}} \leftarrow a_2 \\ \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}} \leftarrow b \end{cases}$$

Therefore :

$$S = \begin{cases} a \rightarrow \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}} b \\ a \rightarrow \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}} \\ b \rightarrow \underbrace{a_1 a_1 \dots a_1}_{k_2 \text{-times}} \underbrace{a_2 a_2 \dots a_2}_{(k_1 - k_2) \text{-times}} \end{cases}$$

Example 12 : $f(n) = 3f(n-1) + 2f(n-2)f(1) = 1$; $f(2) = 1$

$$S = \begin{cases} a_1 \rightarrow a_1 a_1 a_2 b \\ a_2 \rightarrow a_1 a_1 a_2 \\ b \rightarrow a_1 a_1 a_2 \end{cases}$$

$$N_3(a) = 3f(1) = 3 \quad N_4(a) = 15$$

$$N_3(b) = 2f(1) = 2 \quad N_4(b) = 2$$

$N_4(b) = 2$ therefore, in the third stage there are two cells of the type a_1 to obtain $N_4(b) = 2$.

Therefore $L = (\{a_1, a_2, b, s, t\}, s, S)$ where :

$$S = \begin{cases} s \rightarrow t \\ t \rightarrow a_1 a_1 a_2 b b \\ a_1 \rightarrow a_1 a_1 a_2 b \\ a_2 \rightarrow a_1 a_1 a_2 \\ b \rightarrow a_1 a_1 a_2 \end{cases}$$

III.4.3. $r \geq 3$

Generally, starting from the s -th stage we shall keep back the growth on r consecutive stages and we'll set restrictive conditions as in the cases a, b, c from III.4.2.

Example 13 : $f(n) = f(n-1) + 3f(n-2) + f(n-3)f(1) = 1, f(2) = 1, f(3) = 4$

We assume that we need four types of cells (we are going to see that this number is to great one cell remain inactive, therefore three types of cells are enough).

Therefore $I = \{a, b, c, d\}$

We assume that the growth is in the s -th stage

$$\leftarrow a + b + c + \boxed{d} \quad s\text{-th stage}$$

$$\leftarrow 3a + 3b + 3c + \boxed{3d} + a \quad (s+1)\text{-th stage}$$

$$\begin{array}{l} a + b + c + \boxed{d} + b + 3a \\ a + b + c \end{array} \quad (s+2)\text{-th stage}$$

Therefore d is an inactive cell. Therefore, $I = \{a, b, c\}$

$$I = \begin{cases} a \leftarrow a + b + c + d \\ b \leftarrow a \\ c \leftarrow b + 2a \end{cases} \Rightarrow S = \begin{cases} a \rightarrow abcc \\ b \rightarrow ac \\ c \rightarrow a \end{cases}$$

Because $|w_1| = 1$ and $w_1 \Rightarrow w_2$, $|w_2| = 1$ and $|w_3| = 4$ there results that c is the basic cell.

Therefore $L = (\{a, b, c\}, c, S)$

$$S = \begin{cases} a \rightarrow abcc \\ b \rightarrow ac \\ c \rightarrow a \end{cases}$$

A growth of the $\mathcal{L}$ -system L is represented in figure 14.

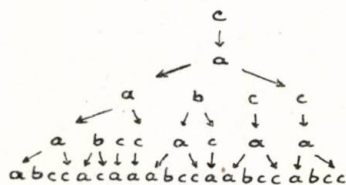


Fig. 14

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Therefore $\lambda = (a, b, c)$.
We assume that the growth is in the s -th stage

$$\begin{aligned} & a + b + c = d + 3a \\ & \rightarrow a + b + c = d \\ & \rightarrow a + b + c = d + 3a \\ & \rightarrow a + b + c = d + 3a \end{aligned}$$

Therefore a is an inactive cell. Therefore $\lambda = (a, b, c)$

$$\begin{aligned} & \left\{ \begin{array}{l} a \rightarrow abca \\ b \rightarrow a \\ c \rightarrow a \end{array} \right. \Rightarrow \lambda = \left\{ \begin{array}{l} a \rightarrow abca \\ b \rightarrow a \\ c \rightarrow a \end{array} \right. \\ & \left\{ \begin{array}{l} a \rightarrow abca \\ b \rightarrow a \\ c \rightarrow a \end{array} \right. \Rightarrow \lambda = \left\{ \begin{array}{l} a \rightarrow abca \\ b \rightarrow a \\ c \rightarrow a \end{array} \right. \end{aligned}$$

Because $w_1 = 1$ and $w_2 = 1$ and $w_3 = 1$ and $w_4 = 1$ there results that
if the basic cell

Therefore $\lambda = (a, b, c)$

$$\left\{ \begin{array}{l} a \rightarrow abca \\ b \rightarrow a \\ c \rightarrow a \end{array} \right. \Rightarrow \lambda = \left\{ \begin{array}{l} a \rightarrow abca \\ b \rightarrow a \\ c \rightarrow a \end{array} \right.$$

growth of the K -system λ is represented in figure 4.



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ON PROPERTIES OF SOME MARTINGALES

ZHUANG XINGWU

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n \in \mathbb{N}})$ be the following stochastic basis: $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, P = the Lebesgue measure on $[0, 1]$, $\mathcal{F}_n = \mathcal{B}\left\{\left[\frac{i}{m^n}, \frac{i+1}{m^n}\right], i = 0, 1, \dots, m^n - 1\right\}$ where $m \geq 2$ is a fixed natural number.

In the following we shall point out some properties of the martingales adapted to this stochastic basis. This paper is written under the guidance of Professor Ion Cuculescu.

We remind that a $(\mathcal{F}_n)$ -martingale (X_n) is singular if $x_n \geq 0$ and $x_n \xrightarrow{a.e.} 0$.

Proposition 1. *Let (x_n) be a nonnegative $(\mathcal{F}_n)$ -martingale then x_n can be written as*

$$x_n = y_1 \cdot y_2 \cdot \dots \cdot y_n$$

where $y_1 = x_1$ and

$$(1) \quad y_n = \sum_{k=0}^{m^{n-1}-1} \sum_{j=1}^m C_{kj}^{(n)} \cdot 1_{\left[\frac{mk+j-1}{m^n}, \frac{mk+j}{m^n}\right]}$$

with

$$(2) \quad 0 \leq C_{kj}^{(n)} \leq m \text{ and } \sum_{j=1}^m C_{kj}^{(n)} = m \text{ for } n \geq 1, 0 \leq k \leq m^{n-1} - 1$$

Proof: Let us write down $y_n = \frac{x_n}{x_{n-1}}$ if $x_{n-1} \neq 0$ and y_n satisfies (1) if $x_{n-1} = 0$ for $n > 1$. Let $x_0 = 0$, then $x_n = x_{n-1} y_n$ for $n \geq 1$ and as $M(x_n | \mathcal{F}_{n-1}) = x_{n-1}$ we have that $M(y_n | \mathcal{F}_{n-1}) = 1$, y_n is $\mathcal{F}_n$ -measurable and $y_n \geq 0$.

Then y_n can be written as in (1), we must only verify (2). Since $y_n \geq 0$ and $M(y_n | \mathcal{F}_{n-1}) = 1$, we have $C_{kj}^{(n)} \geq 0$ and

$$M(y_n) = \sum_{k=0}^{m^{n-1}-1} \sum_{j=1}^m C_{kj}^{(n)} \frac{1}{m^n} = 1$$

Taking into account that $\left[\frac{i}{m^{n-1}}, \frac{i+1}{m^{n-1}}\right] \in \mathcal{F}_{n-1}$, we have

$$\begin{aligned} 1\left[\frac{i}{m^{n-1}}, \frac{i+1}{m^{n-1}}\right] &= M\left(1\left[\frac{i}{m^{n-1}}, \frac{i+1}{m^{n-1}}\right] y_n \mid \mathcal{F}_{n-1}\right) = \\ &= M\left(1\left[\frac{i}{m^{n-1}}, \frac{i+1}{m^{n-1}}\right) \sum_{k=0}^{m^{n-1}-1} \sum_{j=1}^m C_{kj}^{(n)} \cdot 1\left[\frac{mk+j-1}{m^n}, \frac{mk+j}{m^n}\right] \mid \mathcal{F}_{n-1}\right) = \\ &= M\left(\sum_{j=1}^m C_{ij}^{(n)} \cdot 1\left[\frac{mi+j-1}{m^n}, \frac{mi+j}{m^n}\right] \mid \mathcal{F}_{n-1}\right) \end{aligned}$$

So that

$$\frac{1}{m^{n-1}} = M\left(1\left[\frac{i}{m^{n-1}}, \frac{i+1}{m^{n-1}}\right)\right) = M\left(\sum_{j=1}^m C_{ij}^{(n)} \cdot 1\left[\frac{mi+j-1}{m^n}, \frac{mi+j}{m^n}\right]\right) = \frac{\sum_{j=1}^m C_{ij}^{(n)}}{m^n}$$

which means that $\sum_{j=1}^m C_{ij}^{(n)} = m$ for $n \geq 0$, $0 \leq i \leq m^{n-1} - 1$ and so we have (2), q.e.d.

Proposition 2. Let y_n be as in Proposition 1 and

$$B_k^{(n)} = \{C_{kj}^{(n)} \mid j = 1, 2, \dots, m\}$$

If

$$(3) \quad B_0^{(n)} = B_1^{(n)} = \dots = B_{m^{n-1}-1}^{(n)} = B^{(n)}$$

for every $n \geq 1$, then (y_n) are independent random variables.

Proof: It is sufficient to prove that $y_1, y_2, \dots, y_n$ are independent. For every $a_i \in B^{(i)}$, we will prove that

$$\begin{aligned} (4) \quad P(y_1 = a_1, y_1 = a_2, \dots, y_n = a_n) &= \\ &= P(y_1 = a_1) \cdot P(y_2 = a_2) \dots P(y_n = a_n) \end{aligned}$$

According to (3) $\{y_n = a_n\}$ is a union of intervals of the same length $\left(\frac{g^{(n)}}{m^n}\right)$ where $g^{(n)}$ is the number of elements of $B^{(n)}$, which number is equal to a_n , and there are m^{n-1} such intervals, let us denote them by $Q_k^{(n)}$ ($k = 0, 1, 2, \dots, m^{n-1} - 1$) so we have $P(\{y_n = a_n\}) = \frac{g^{(n)}}{m^n} \cdot m^{n-1} = \frac{g^{(n)}}{m}$. According to (3) $\{y_1 = a_1, y_2 = a_2, \dots, y_{n-1} = a_{n-1}\}$ is a union of

intervals of the same length $\frac{1}{m^{n-1}}$ and each of these intervals contains an unique $Q_k^{(n)}$, we have

$$P(y_n = a_n | y_1 = a_1, y_2 = a_2, \dots, y_{n-1} = a_{n-1}) = \frac{q^{(n)}/m^n}{1/m^{n-1}} = \frac{g^{(n)}}{m},$$

hence

$$\begin{aligned} P(y_1 = a_1, y_2 = a_2, \dots, y_n = a_n) &= P(y_n = a_n | y_1 = a_1, y_2 = a_2, \dots, y_{n-1} = a_{n-1}) \\ &\quad \cdot P(y_1 = a_1, y_2 = a_2, \dots, y_{n-1} = a_{n-1}) = \\ &= P(y_n = a_n) \cdot P(y_1 = a_1, y_2 = a_2, \dots, y_{n-1} = a_{n-1}) \end{aligned}$$

and, step by step, (4) follows, q.e.d.

Lemma 1. If we denote $A = \{1, 2, \dots, m\}$, $I^{(n)} = \{(i_1, i_2, \dots, i_n) | i_j \in A, j = 1, 2, \dots, n\}$

$P_k^{(n)}(M) = \{i_k | (i_1, i_2, \dots, i_k, \dots, i_n) \in M, M \subset I^{(n)}\}$ then $I^{(n)}$ can be divided into the following m^{n-1} sets

$$I^{(n)} = \bigcup_{j=1}^{m^{n-1}} I_j^{(n)}, I_{j_1}^{(n)} \cap I_{j_2}^{(n)} = \Phi \quad (j_1 \neq j_2),$$

where $I_j^{(n)}$ contains m elements of $I^{(n)}$ and $P_k^{(n)}(I_j^{(n)}) = A$ for $k = 1, 2, \dots, n$; $j = 1, 2, \dots, m^{n-1}$.

Proof: We shall use induction. It is obvious for $n = 1$. If it is verified for $n - 1$, we shall prove that it is valid for n . According to the hypothesis of induction $I^{(n)}$ can be expressed as

$$I^{(n)} = \bigcup_{j=1}^{m^{n-1}} \{(b_j^{(n-1)}, i_n) | b_j^{(n-1)} \in I_j^{(n-1)}, i_n \in A\}$$

where $I_{j_1}^{(n-1)} \cap I_{j_2}^{(n-1)} = \Phi$ ($j_1 \neq j_2$), $I_j^{(n-1)}$ contains m elements of $I^{(n-1)}$ and $P_k^{(n-1)}(I_j^{(n-1)}) = A$ for $k = 1, 2, \dots, n - 1$; $j = 1, 2, \dots, m^{n-2}$. For the sake of simplicity we denote

$$I_j^{(n-1)} = \{d_1^{(j)}, d_2^{(j)}, \dots, d_m^{(j)}\} \quad (j = 1, 2, \dots, m^{n-2})$$

then

$$\begin{aligned} \{(b_j^{(n-1)}, i_n) | b_j^{(n-1)} \in I_j^{(n-1)}, i_n \in A\} &= \{(d_i^{(j)}, i_n) | d_i^{(j)} \in I_j^{(n-1)}, i \in A, i_n \in A\} \\ &= \bigcup_{i=1}^m \{(d_1^{(j)}, i), (d_2^{(j)}, i + 1), \dots, (d_{m-i+1}^{(j)}, m), (d_{m-i+2}^{(j)}, 1), \dots, (d_m^{(j)}, i - 1)\} = \\ &= \bigcup_{i=1}^m D_i^{(j)} \end{aligned}$$

where

$$D_i^{(j)} = \{(d_1^{(j)}, i), (d_1^{(j)}, i+1), \dots, (d_{m-i+1}^{(j)}, m), (d_{m-i+2}^{(j)}, 1), \dots, (d_m^{(j)}, i-1)\}$$

$$D_{i_1}^{(j)} \cap D_{i_2}^{(j)} = \Phi \quad (i_1 \neq i_2)$$

$D_i^{(j)}$ contains m elements of $I^{(n)}$

and $P_k^{(n)}(D_i^{(j)}) = A$ for $k = 1, 2, \dots, n$.

So that

$$I^{(n)} = \bigcup_{j=1}^{m^{n-2}} \bigcup_{i=1}^m D_i^{(j)} = \bigcup_{j=1}^{m^{n-1}} I_j^{(n)}$$

where $I_{j_1}^{(n)} \cap I_{j_2}^{(n)} = \Phi$ ($j_1 \neq j_2$), $I_j^{(n)}$ contains m elements of $I^{(n)}$ and $P_k^{(n)}(I_j^{(n)}) = A$ for $k = 1, 2, \dots, n$; $j = 1, 2, \dots, m^{n-1}$. q.e.d.

Lemma 2. Let $C_j^{(i)}$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$) be nonnegative numbers, $\vec{j} = (j_1, j_2, \dots, j_n)$ ($j_i = 1, 2, \dots, m$; $i = 1, 2, \dots, n$), $\Pi_{\vec{j}} =$

$$= \prod_{i=1}^n C_{j_i}^{(i)} \text{ and } M_1 = \{\vec{j} \mid \Pi_{\vec{j}} \leq \sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}}\}, M_2 = \{\vec{j} \mid \Pi_{\vec{j}} \geq \sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}}\}$$

then $\overline{M}_1 \geq m^{n-1}$, $\overline{M}_2 \geq m^{n-1}$ for $n \geq 1$ where $\overline{M}_i$ means cardinal number of set M_i ($i = 1, 2$).

Proof.: Using Lemma 1. We see that $\{\Pi_{\vec{j}} \mid \vec{j} = (j_1, j_2, \dots, j_{n-1}), j_i = 1, 2, \dots, m; i = 1, 2, \dots, n-1\} = \{C_{i_1}^{(1)} C_{i_2}^{(2)} \dots C_{i_{n-1}}^{(n-1)} \mid i_k = 1, 2, \dots, m, k = 1, 2, \dots, n-1\}$

can be divided into m^{n-2} sets such that every set has m elements $C_{i_1}^{(1)} C_{i_2}^{(2)} \dots$

$\dots C_{i_{n-1}}^{(n-1)}$ ($l = 1, 2, \dots, m$) and $\{i_j^l \mid l = 1, 2, \dots, m\} = \{1, 2, \dots, m\}$ for every $j = 1, 2, \dots, n-1$. Let us take one of these m^{n-2} sets and denote $b_l = C_{i_1}^{(1)} C_{i_2}^{(2)} \dots C_{i_{n-1}}^{(n-1)}$ ($l = 1, 2, \dots, m$). Without lack of generality we can suppose $b_1 \leq b_2 \leq \dots \leq b_m$ and $C_{j_1}^{(n)} \leq C_{j_2}^{(n)} \leq \dots \leq C_{j_n}^{(n)}$ where $(j_1, j_2, \dots, j_n)$ is some permutation of $(1, 2, \dots, n)$. As $b_1 C_{j_m}^{(n)} \cdot b_2 C_{j_{m-1}}^{(n)} \dots b_m C_{j_1}^{(n)} = \prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}$, we conclude that there is at least a subproduct $b_i C_{j_{m-i+1}}^{(n)}$

satisfying the inequality

$$b_i C_{j_{m-i+1}}^{(n)} \leq \sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}}$$

then $b_1 C_{j_{m-i+1}}^{(n)}, b_2 C_{j_{m-i+1}}^{(n)} \dots b_{i-1} C_{j_{m-i+1}}^{(n)}, b_i C_{j_{m-i}}^{(n)}, \dots, b_i C_{j_1}^{(n)}$

all of them are less than $\sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}}$. Each of the m^{n-2} sets with $C_1^{(n)}, C_2^{(n)}, \dots, C_m^{(n)}$ produces at least n different elements which belong to

$$\{\Pi_{\vec{j}} \mid \vec{j} = (j_1, j_2, \dots, j_n), j_i = 1, 2, \dots, m; i = 1, 2, \dots, n\}$$

and are less than $\sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}}$, hence $\overline{M}_1 \geq m^{n-1}$ for $n > 1$

Similarly we can prove that $\overline{M}_2 \geq m^{n-1}$, q.e.d.

Proposition 3. Let (x_n) is a nonnegative $(\mathcal{F}_n)$ -martingale and suppose that (3) is satisfied. In (3) we denote

$$B^{(n)} = (C_1^{(n)}, C_2^{(n)}, \dots, C_m^{(n)}) \text{ for } n \geq 1, \text{ then}$$

$$\prod_{i=1}^{\infty} \prod_{j=1}^m C_j^{(i)} = 0 \Leftrightarrow (x_n) \text{ is singular}$$

Proof. " $\Rightarrow$ ", According to Proposition 1 and (3), we have

$$\begin{aligned} x_n &= y_1 y_2 \dots y_n = \\ &= \sum_{1 \leq i_1, i_2, \dots, i_n \leq n} C_{i_1}^{(1)} C_{i_2}^{(2)} \dots C_{i_n}^{(n)} \cdot 1_{B_{(i_1, i_2, \dots, i_n)}} \end{aligned}$$

where $B_{(i_1, i_2, \dots, i_n)} \cap B_{(i'_1, i'_2, \dots, i'_n)} = \Phi$ for $(i_1, i_2, \dots, i_n) \neq (i'_1, i'_2, \dots, i'_n)$

and $P(B_{(i_1, i_2, \dots, i_n)}) = \frac{1}{m^n}$ for $1 \leq i_1, i_2, \dots, i_n \leq m$

We denote $Q^{(n)} = \left\{ (i_1, i_2, \dots, i_n) \mid C_{i_1}^{(1)} C_{i_2}^{(2)} \dots C_{i_n}^{(n)} \leq \sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}} \right\}$

$$A^{(n)} = \bigcup_{(i_1, i_2, \dots, i_n) \in Q^{(n)}} B_{(i_1, i_2, \dots, i_n)}$$

$$A_{\varepsilon}^{(n)} = \{\omega \mid x_n \leq \varepsilon\} \quad \text{where } \varepsilon > 0$$

First by using Lemma 2 we have $P(A^{(n)}) \geq \frac{1}{m}$ for $n \geq 1$. According

to the condition $\prod_{i=1}^{\infty} \prod_{j=1}^m C_j^{(i)} = 0$, for any $\varepsilon > 0$ there exists N such that

$\sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}} \leq \varepsilon$ for $n > N$. It follows that $A^{(n)} \subset A_{\varepsilon}^{(n)}$ and $P(A_{\varepsilon}^{(n)}) \geq \frac{1}{m}$ for $n > N$.

Now we shall prove that $P\left(\left\{\prod_{i=1}^{\infty} y_i < \infty\right\}\right) = 1$. Thanks to the

fact that $\left\{\prod_{i=1}^{\infty} y_i < \infty\right\}$ is a tail event (namely, $\left\{\prod_{i=1}^{\infty} y_i < \infty\right\} \in \bigcup_{k=1}^{\infty} \mathcal{B}(\mathcal{F}_k \cap \mathcal{F}_{k+1} \cap \dots)$, see [1] p. 27) and that (y_n) is a sequence of indepen

dent random variables, by the Law 0-1, we have $P\left(\left\{\prod_{i=1}^{\infty} y_i < \infty\right\}\right) = 0$ or 1. We show that $P\left(\left\{\prod_{i=1}^{\infty} y_i < \infty\right\}\right) > 0$. Otherwise $P\left(\left\{\prod_{i=1}^{\infty} y_i = \infty\right\}\right) = 1$, namely, $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \prod_{i=1}^n y_i = \infty$ almost everywhere. According to the Egorov Theorem, there exists A such that $P(A) < \frac{1}{m}$, $\lim_{n \rightarrow \infty} x_n = \infty$ uniformly on $\Omega - A$. Hence for every $\varepsilon > 0$ there exists N_1 such that $x_n > \varepsilon$, on $\Omega - A$ for any $n > N_1$. Consequently $P(A_\varepsilon^{(n)}) = P(\{\omega \mid x_n \leq \varepsilon\}) \leq P(A) < \frac{1}{m}$ for $n > N_1$, which is contradictory to the conclusion before. Therefore

$$P\left(\left\{\prod_{i=1}^{\infty} y_i < \infty\right\}\right) = 1$$

Further we shall prove that $P\left(\left\{\prod_{i=1}^{\infty} y_i = 0\right\}\right) \geq \frac{1}{m}$. Otherwise, $P\left(\left\{\prod_{i=1}^{\infty} y_i > 0\right\}\right) > \frac{m-1}{m}$. According to the Lusin Theorem, for any $\delta > 0$, exists a closed set F such that $F \subset \left\{\prod_{i=1}^{\infty} y_i \geq 0\right\} = E$, $P(E - F) < \delta$ and $\prod_{i=1}^{\infty} y_i$ is continuous on F . By properties of continuous function $\min_{\omega \in F} \prod_{i=1}^{\infty} y_i = q > 0$. Let $0 < \varepsilon < q$, exists N_2 such that $\prod_{i=1}^n y_i > \varepsilon$ on F for $n > N_2$, then $A_\varepsilon^n \subset \Omega - F$ for $n > N_2$. If we take $\delta > 0$ such that $0 < \delta < P\left(\left\{\prod_{i=1}^{\infty} y_i > 0\right\}\right) - \frac{m-1}{m}$, then $P(F) = P(E) - P(E - F) > P(E) - \delta > \frac{m-1}{m}$. It results that $P(A_\varepsilon^{(n)}) \leq P(\Omega - F) < 1 - \frac{m-1}{m} = \frac{1}{m}$ for $n > N_2$, which is contradictory to the conclusion before.

$\left\{\prod_{i=1}^{\infty} y_i = 0\right\}$ too is a tail event and (y_n) is a sequence of independent random variables, by the Law 0-1 we have

$$P\left(\left\{\prod_{i=1}^{\infty} y_i = 0\right\}\right) = 1$$

Namely, $x_n = y_1 y_2 \dots y_n \rightarrow 0$ for almost everywhere.

„ $\Leftarrow$ ” Contrarily we suppose $\lim_{n \rightarrow \infty} \prod_{i=1}^n \prod_{j=1}^m C_j^{(i)} \neq 0$. As $\prod_{j=1}^m C_j^{(i)} \leq \left(\frac{\sum_{j=1}^m C_j^{(i)}}{m} \right)^m = 1$, $\left\{ \prod_{i=1}^n \prod_{j=1}^m C_j^{(i)} \right\}_{n \in \mathbb{N}}$ is a decreasing sequence, then exist $\alpha > 0$ such that $\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)} \geq \alpha$ for $n \geq 1$ and

$$\lim_{n \rightarrow \infty} \prod_{i=1}^n \prod_{j=1}^m C_j^{(i)} = \alpha$$

We denote

$$\bar{Q}^{(n)} = \left\{ (i_1, i_2, \dots, i_n) \mid C_{i_1}^{(1)} C_{i_2}^{(2)} \dots C_{i_n}^{(n)} \geq \sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}} \right\}$$

$$\begin{aligned} \bar{A}^{(n)} &= \bigcup_{(i_1, i_2, \dots, i_n) \in \bar{Q}^{(n)}} B_{(i_1, i_2, \dots, i_n)} \cap B_{(i'_1, i'_2, \dots, i'_n)} = \\ &= \emptyset \quad ((i_1, i_2, \dots, i_n) \neq (i'_1, i'_2, \dots, i'_n)) \end{aligned}$$

$$\bar{A}_1^{(n)} = \left\{ \omega \mid x \geq \frac{1}{\alpha^m} \right\}$$

By Lemma 2 we have $P(\bar{A}^{(n)}) \geq \frac{1}{m}$ for $n \geq 1$. As

$$\sqrt[n]{\prod_{i=1}^n \prod_{j=1}^m C_j^{(i)}} \geq \sqrt[n]{\alpha} \text{ for } n \geq 1, \text{ then } \bar{A}^{(n)} \subset \bar{A}_1^{(n)}, \text{ hence}$$

$P\left(\bar{A}_1^{(n)}\right) \geq \frac{1}{m}$ for $n \geq 1$. This means that x_n does not converge to null, which is contradictory to the given condition.

The proof of Proposition is completed.

Corollary 1. Let (x_n) be as in Proposition 1. If for any $i \geq 1$, $C_j^{(i)} \equiv C_j$ ($j = 1, 2, \dots, m$), then (x_n) is singular* if and only if $C_j \neq 1$ for $j = 1, 2, \dots, m$.

Proof. : If $C_j \neq 1$ for $j = 1, 2, \dots, m$. We remind that the inequality $\prod_{j=1}^m C_j \leq \left(\sum_{j=1}^m C_j / m \right)^m$ becomes equality if and only if $C_1 = C_2 = \dots = C_m \geq 0$. Since $0 \leq C_j \leq m$ and $\sum_{j=1}^m C_j = m$, there exist i_1, i_2 such that $C_{i_1} \neq C_{i_2}$ hence $\sum_{j=1}^m C_j < \left(\sum_{j=1}^m C_j / m \right)^m = 1$, and $\prod_{i=1}^{\infty} \prod_{j=1}^m C_j^{(i)} = 0$. With aid of Proposition 3, (x_n) is singular.



Conversely, if (x_n) is singular, by Proposition 3, we have $\lim_{n \rightarrow \infty} \left(\prod_{j=1}^n C_j \right)^n = 0$, hence $\prod_{j=1}^m C_j < 1$, and it results $C_j \neq 1$ for $j=1, 2, \dots, m$. q.e.d.

Corollary 2: When $m = 2$ we have the following results:

1. If (x_n) is nonnegative $\mathcal{F}_n$ -martingale then x_n can be written as

$$x_n = y_1 \cdot y_2 \cdot \dots \cdot y_n$$

where

$$y_1 = x_1$$

$$y_n = \sum_{k=0}^{n-1} \sum_{i=1}^m \left[C_k^{(n)} \cdot 1 \left(\frac{2k}{2^n}, \frac{2k+1}{2^n} \right) + (2 - C_k^{(n)}) \cdot 1 \left(\frac{2k+1}{2^n}, \frac{2k+2}{2^n} \right) \right]$$

and

$$0 \leq C_k^{(n)} \leq 2 \text{ for } n \geq 1, 0 \leq k \leq 2^{n-1} - 1$$

2. If $C_k^{(n)} = C^{(n)}$ for $0 \leq k \leq 2^{n-1} - 1$, then (x_n) is a sequence of independent random variables.

3. If (x_n) is nonnegative $\mathcal{F}_n$ -martingale and $C_k^{(n)} \equiv C^{(n)}$ for $0 \leq k \leq 2^{n-1} - 1$, then (x_n) is singular if and only

$$\prod_{n=1}^{\infty} C^{(n)} (2 - C^{(n)}) = 0$$

4. If (x_n) is a nonnegative $\mathcal{F}_n$ -martingale and $C^{(n)} \equiv C$ for $n \geq 1$ then (x_n) is singular if and only if $C \neq 1$

5. If (x_n) is a nonnegative $\mathcal{F}_n$ -martingale and

$$\prod_{n=1}^{\infty} [\max_k C_k^{(n)} (2 - \min_k C_k^{(n)})] = 0 \quad \text{then } (x_n) \text{ is singular.}$$

6. If (x_n) is a nonnegative $\mathcal{F}_n$ -martingale and

$$\prod_{n=1}^{\infty} [\min_k C_k^{(n)} (2 - \max_k C_k^{(n)})] > 0 \quad \text{then } (x_n) \text{ is not singular.}$$

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VIAȚA FACULTĂȚII

75 DE ANI DE LA NAȘTEREA PROFESORULUI GR. C. MOISIL

În ziua de 10 Ianuarie 1981, în Amfiteatrul Spiru Harel al Facultății de Matematică a Universității din București a avut loc comemorarea a 75 de ani de la nașterea Profesorului Gr. C. Moisil. Reproducem mai jos textele conferințelor prezentate la această sesiune.

Onorat Prezidiu,

Mult stimați și dragi participanți,

Cu adîncă emoție deschid lucrările Sesiunii Științifice Comemorative cu ocazia împlinirii a 75 de ani de la nașterea marelui fiu al neamului nostru, academicianul profesor Grigore C. Moisil.

Cu adîncă emoție, deoarece și eu mă număr printre elevii profesorului Moisil. Sub îndrumarea sa mi-am făcut ucenicia în cercetarea matematică, în seminarul profesorului Moisil din anii 49—50 am realizat primele lucrări, sub îndrumarea sa am ținut primele mele cursuri.

Avea profesorul Moisil un dar neîntrecut de a intui liniile mari de dezvoltare ale matematicii. Un puternic suflu novator străbătea întreaga sa activitate. Știa să entuziasmeze tineretul și să-l îndrepte spre domenii noi, neexplorate. Știa să fie exigent și în acelaș timp să încurajeze. Avea încredere în tineret și nu ezita să-i încredințeze răspunderi mari, iar tineretul i s-a alăturat întotdeauna cu elan. Mărturie stau școlile matematice, pe care le-a creiat în domenii atît de variate: logică matematică, algebră, ecuații cu derivate parțiale, mecanică, teoria algebrică a mecanismelor automate, școli care au dus mai departe importanțele sale cercetări și au dezvoltat sub impulsul său noi direcții, școli care au înscris o pagină de glorie a matematicii românești.

Profesorul Moisil a luptat cu perseverență și energie pentru a creia în țara noastră acea înțelegere a rolului matematicii în etapa actuală și în viitor, înțelegere care să-i asigure condițiile de dezvoltare și de aplicare. Cu neasemuită putere de previziune, el a sesizat importanța informaticii, ciberneticii și automaticii, și a desfășurat o acțiune neobosită pentru a pune bazele învățămîntului și cercetării matematice în aceste domenii. A creiat centre de calcul, a propagat noile instrumente matematice în

tehnică și pe de altă parte a stimulat pătrunderea matematicii în toate celelalte științe : chimie, biologie, economie, științele umaniste, lingvistică, istorie, arheologie, artă. Cu inteligența sa scilicitoare, cu verva neîntrecută, cu uriașa sa putere de convingere, prin scris, prin conferințe, prin organizare de nenumărate manifestări științifice a făcut cunoscute în țara noastră aceste trăsături specifice ale matematicii contemporane.

Ne-a învățat să fim exigenți cu noi înșine și cu prestigiul matematicii românești, cu prestigiul facultății și al universității noastre. Ne-a învățat prin fapta și exemplul său să iubim matematica, să-i dăruim tot timpul nostru. Care dintre noi ar putea uita frumosul aforism, din care citez doar începutul : „Matematica se face douăzeci și patru de ore pe zi...”

Rolul său în dezvoltarea învățămîntului și cercetării matematice în facultatea noastră îndeosebi, unde a fost profesor vreme de peste trei decenii, la institutul de matematică din București, ca și dealtfel pe plan național a fost hotărîtor.

„Anii trec ca apa” spune un vers dintr-o cunoscută baladă al lui Topîrceanu, în care în sinul mut al naturii se pierde tot sbuciumul unei vieți omenеști. Profesorul Moisil dezmente total această lege implacabilă. Cu trecerea timpului opera sa matematică devine mai actuală, mai impresionantă, dobîndește o răspîndire mai largă, o recunoaștere mai deplină (dovadă rubrica consacrată în ultimii ani algebrei Lukasivici—Moisil în *Mathematical Reviews*), școlile pe care le-a fondat rodesc și se dezvoltă, cuvîntul său devine tot mai convingător, adevărurile pentru care a luptat se impun tuturor, figura sa intrată în legendă se împletește cu realitatea. Pentru noi toți e viu, vorbele sale, timbrul vocii sale ne răsună în inimă. Mi s-ar părea foarte firesc să deschidă ușa acestui amfiteatru, ca de obicei la adunările facultății, să se așeze încet în primul rînd, să scoată un ziar din geanta plină de lucrări, aflată alături pe bancă, să ne citească cîteva rînduri și, plecînd de la ele, să interpreteze imperativul zilei de azi prin perspectiva zilelor de mîine, în care matematica va fi chemată să înfăptuiască atitea. Mi s-ar părea foarte firesc, deschizînd de data aceasta eu ușa sălilor 1 sau 2 (după numele lor de odinioară), să-l revăd la catedră explicînd în stilul său plin de originalitate, într-un limbaj ce se întipărește pentru totdeauna în mintea auditoriului, vreuna din nenumăratele fațete ale matematicii, pe care o concepea unitar în toată vastitatea ei, ca un adevărat demiurg al renașterii.

Aș dori să pot exprima aici întreaga recunoștință pe care i-o port și totodată să aduc un sincer omagiu distinsei sale soții, căreia îi datorăm volume atît de emoționante, consacrate profesorului nostru drag.

Și aș dori să închei cu cuvintele pline de optimism ale profesorului Moisil din medalionul, pe care l-am urmărit de curînd la Televiziune :

„Speranța că vom ști să fie totul mai bine”,
cuvinte care închid un luminos îndreptar pentru întreaga noastră activitate.

Prof. C. ANDREIAN CAZACU

Astăzi, 10 ianuarie 1981, GRIGORE MOISIL, dacă ar fi fost în viață, ar fi fost în mijlocul nostru și noi l-am fi sărbătorit cum făceam în fiecare an. Dar, Moisil nu mai e printre noi. Noi însă, nu l-am uitat,

după cum nu-l vor uita nici generațiile de matematicieni care vor veni după noi.

În general, matematicienii nu sînt cunoscuți și apreciați decît în cercul restrîns al confrăților lor. Grigore Moisil a făcut excepție de la această regulă, numărul celor care l-au cunoscut și admirat depășind cu mult lumea matematică. Este drept însă, că notorietatea lui nu se datora numai matematicii. Om de mare cultură literară și filozofică, Moisil era un conferențiar strălucit, cucerind prin verva, inteligența și spontaneitatea lui. Era la televiziune unul dintre cei mai simpatizați vorbitori. A scris în ziare și reviste un număr mare de articole, în care, sub o formă agreabilă, dezbătea cu competență probleme de mare interes științific. La toate acestea, se adăuga purtarea lui originală, plină de farmec, și cuvintele lui de spirit, cu o largă circulație, care au contribuit, nu cu puțin, la marea lui popularitate.

La comemorarea vîrstei de 75 ani, trebuie însă să punem pe prim plan realizările majore din timpul vieții lui și să-l evocăm pe Moisil în adevărata lui lumină ca pe un savant care a contribuit la progresul științei românești.

Moisil a început să lucreze în matematici încă din liceu, în calitate de colaborator al Gazetei matematice, distingîndu-se prin preocupări ce depășeau cu mult nivelul cunoștințelor căpătate în școala secundară din acel timp. El dovedea, încă de pe atunci, o mare ușurință în asimilarea de noi cunoștințe matematice, însușire care l-a caracterizat dealtminteri, în tot timpul activității lui.

Doctoratul în matematici l-a susținut la Universitatea din București, în anul 1929, la vîrsta de numai 23 ani, cu o teză remarcabilă intitulată „Mecanica analitică a sistemelor continue”. Ulterior, aplicînd aceeași metodă și în geometria diferențială, precum și în alte domenii, Moisil se înscrie printre precursorii importante discipline matematice de astăzi, analiza funcțională. Aceste prime lucrări de matematici superioare, au atras atenția ilustrului matematician italian Vito Volterra asupra lui. Moisil avea calea deschisă pentru a continua cercetările începute cu succes și a deveni un specialist în analiza funcțională. Dar firea lui se împotriva acestui lucru. El nu putea rămîne cantonat într-un domeniu. O curiozitate științifică ieșită din comun l-a minat să studieze cît mai mult din toată matematica și să dăruiască științei peste 300 lucrări din cele mai variate domenii: analiza matematică, geometrie, algebră, mecanică, logică matematică, teoria probabilităților, fundamentele matematicii, teoria algebrică a mecanismelor automate, lingvistică matematică, informatică etc. În unele din aceste domenii, Moisil a fost nu numai un deschizător de drumuri, fiind primul în țară care a publicat în unele domenii lucrări originale dar, grupînd tineri în jurul lui, a creat și școli, ca de exemplu în logica matematică sau teoria algebrică a mecanismelor automate.

Într-un articol intitulat „Blaise Pascal și Dl Joseph Bertrand” Anatole France compară între ele două figuri ale matematicii: Blaise Pascal și Wilhelm Leibnitz. Citindu-l pe Bertrand, Anatole France scrie „Cu aceiași profunzime și egală aptitudine, spiritele lor erau neasemănătoare. Leibnitz, curios de toate, exceptînd detaliile, propune metode noi,

lăsînd altora grija și onoarea de a le aplica. Pascal, dimpotrivă, vrea să precizeze totul : numai rezultatele îl interesează. Leibnitz descoperă pomul, îl descrie și se depărtează. Pascal arată fructele fără ca să spună originea lor”.

Pentru că în matematică există stiluri, ca și în muzică și poezie, Grigore Moisil, deschizătorul de drumuri, partizanul ideilor noi și generale în știință, recunoaștem imediat că a mers pe stilul lui Leibnitz. Dar cine în matematica română, ar putea fi comparat cu polul opus, după Bertrand, lui Leibnitz ? Aș îndrăzni să afirm că a fost Dan Barbilian. Acest matematician înnăscut, era în stare să dedice un timp îndelungat unei probleme particulare, puțin însemnată din punct de vedere științific, la care aplica toată arta și talentul lui de neîntrecut meșteșugar, pentru a scoate din ea rezultate neprevăzute. Moisil căuta în matematică esențialul și generalul, Barbilian era atras de frumusețea ei. Unul era filozof, celălalt poet.

În sfîrșit, trebuie să amintim că Moisil a fost un mare profesor. Nu profesor în sensul de a expune clar și metodic cursuri clasice. Desigur, are utilitate și acest gen de profesor, mai ales pentru elevii care au o capacitate de înțelegere mai redusă. Ci profesor în sensul înalt al cuvîntului, omul de știință experimentat, care atrage în jurul lui tinerii dotați și îi îndrumază. Moisil, pînă la o vîrstă înaintată se entuziasma în fața noutăților științei, entuziasm pe care îl transmitea și elevilor săi. Nimeni în țara noastră n-a scris mai mult despre matematizarea celorlalte științe și n-a pledat mai mult pentru răspîndirea calculului electronic. Moisil a fost primul profesor director al Centrului de calcul al Universității și președinte al Societății române de matematici.

Moisil a fost membru al Academiei Republicii Socialiste România încă din 1948, cînd această instituție a fost reorganizată. A fost unul dintre cei mai activi academicieni. A fost șeful Secției de algebră aplicată a Institutului de matematică al Academiei, încă de la înființarea acesteia. A fost președintele Comisiei de automatizare din cadrul Academiei și al Comisiei de cibernetică din aceeași instituție. În această calitate a organizat în fiecare an sesiuni de comunicări științifice. Mulți ani a fost membru al Prezidiului Academiei.

La această zi solemnă, Academia Republicii Socialiste România aduce prin mine, omagiul său unuia dintre cei mai vrednici și străluciți membri ai săi.

Acad. GHEORGHE MIHOC

Personalitatea științifică a acad. Gr. C. Moisil

1. Acad. Grigore Moisil reușește să rămînă într-o actualitate de prim plan chiar și pentru cei care nu l-au cunoscut. Cu trecerea timpului Personalitatea sa puternică, afirmată adesea prin intervenții vijelioase, aducătoare de reușite neașteptate, temute de cei care i se opuneau, devine din ce în ce mai admirată ca un simbol al luptătorului neobosit, care nu încetează lupta decît învingător. Și Grigore Moisil a fost într-adevăr

un vajnic luptător pentru promovarea noului și a oamenilor noi în matematica românească.

2. Matematician de valoare internațională s-a făcut cunoscut de la început prin originalitatea gândirii sale îndrăznețe, capabilă să acopere arii largi ale cunoașterii. Doctor în matematică al Universității din București în 1929 printr-o teză intitulată „Mecanica analitică a sistemelor”, se situează printre precursorii analizei funcționale, ca elev al marelui Vito Volterra. Dar curiozitatea sa neobosită îl duce în domeniul ecuațiilor cu derivate parțiale, unde pune bazele unei teorii preliminare a sistemelor liniare cu derivate parțiale. Trece apoi la derivata areolară căreia îi dă o extensiune în spațiile n -dimensionale și la generalizarea ecuațiilor de tip parabolic.

3. Descoperind logica matematică, devine repede o autoritate în acest domeniu, pe care ulterior îl restructurează, dînd o expunere sistematică algebrei logicii, introducînd noțiunile de ideal și studiînd logica modală, logicile trivalente și tetravalente, punînd consecvent în evidență structurile algebrice ale acestora.

Aceasta l-a condus la Teoria algebrică a mecanismelor automate în care a creat o puternică școală românească de cercetători care au adus contribuții foarte valoroase în domeniul informaticii, începînd din 1953, adică în epoca în care această nouă ramură a științei teoretice și aplicative își lua avînturile pe care le cunoaștem azi.

În acest domeniu Acad. Gr. Moisil, cu puterea sa de sinteză a unor domenii ce păreau altora dispartate, a folosit într-o largă măsură lucrările sale din domeniile logicii matematice și algebrei moderne.

Reluînd ideea lui Sestacov de a considera calculul schemelor cu contacte care funcționează în mecanismele automate ca o aplicație a algebrei booleene, Gr. Moisil abordează descrierea funcționării în timp a acestor scheme cu contacte, ceea ce constituie un model dinamic al fenomenelor ce au loc în mecanismele automate. În același timp, el introduce și variabile de stare, pentru a modela fenomenele dinamice prin ecuații de recurență. Cu puterea sa de pătrundere a esenței modelelor, el ia în considerație variabile de stare a contactelor, dar și acele variabile care descriu starea curenților din relele.

În viziunea sa care este fizică acest sistem va pune în evidență funcțiile de lucru ale releelor. Dar această viziune care poate ar fi fost suficientă altora nu satisfăcea complet modelul conceput de el și de aceea un al doilea sistem va cuprinde caracteristicile releelor. Moisil este deci cel care a introdus ecuațiile caracteristice ale releelor.

Ca un strateg călit în desfășurarea bătăliilor pe fronturi largi, el pune în funcțiune un arsenal rafinat și complex format din algebra booleanp, imaginile lui Galois și logicile polivalente după cerințele frontului și stării bătăliei.

Avînd clară concepția adecvării modelelor la complexitatea fenomenelor reale, Gr. Moisil urmărește funcționarea schemelor prin *analiza funcționării lor în diferite scheme*: cu contacte reale, cu relele polarizate cu contacte multipoziționale, cu elemente-ventil. Ca urmare a acestor studii de cazuri concrete, Moisil arată că nu orice schemă poate fi ameliorată în sensul introducerii de circuite suplimentare fără introducerea de relele suplimentare.

Opera sa în domeniul teoriei algebrice a mecanismelor automate, concretizată într-un volum de mare întindere i-a adus premiul de stat pe anul 1962, ca o încununare a operei sale și a școlii pe care a creat-o. Acest volum s-a tradus în limbile rusă și cehă aducând școlii românești de teoria algebrică a mecanismelor automate consacrarea ca a treia școală după cele din SUA și URSS în 1963.

4. Ca deschizător de drumuri s-a afirmat de asemenea și în lingvistica matematică, unde s-a ocupat de traducerea automată prin calculator, de modele logice ale limbii, de indexarea și rezumarea lucrărilor publicate. Școala românească de lingvistică matematică, recunoscută ca și cea de teoria algebrică a mecanismelor automate pe plan internațional, este opera sa.

5. Grigore Moisil începe bătălia pentru înființarea Centrului de Calcul al Universității din București, pe care o câștigă în 1962, pe plan mondial. Acest Centru, care nu a dispus decât peste 2—3 ani de un calculator electronic românesc CIFA 3, de prima generație, construit la Institutul de Fizică Atomică, a pregătit în așteptare un efectiv puternic și entuziast de cercetători în care intrau matematicieni, ingineri, fizicieni, logicieni, etc.

Acest efectiv a reușit ca în câteva săptămâni să stăpânească tehnica de calcul a primului calculator polonez ODRA adus în expoziție la Centrul de Calcul, iar în 1966 să devină în câteva luni specialiști în programarea calculatorului IBM 360/30 care era atunci o noutate din generația III, adus de asemenea în expoziție la Centrul de Calcul unde de altfel a rămas până astăzi.

Efectivul pregătit de Acad. Moisil a fost repartizat în calitate de conducători ai informaticii românești la Institutul Central de Informatică și la numeroase centre de calcul guvernamentale.

Acad. Grigore Moisil a cunoscut gloria pe toate meridianele, fiind invitat să-și expună lucrările în cele mai prestigioase centre științifice, Paris, Roma, Bologna, Firenze, Stanford, Chicago, New-York, Detroit, Moscova, Leningrad, Berlin, Helsinki, Stockholm, Cambridge, Varșovia, Bruxelles, Sofia, Atena, Ankara, Istanbul și altele.

A fost un animator neîntrecut și iubit al tineretului matematic, pe care îl strângea în jurul lui în mai multe colective mixte, după strategia pe care o considera utilă pentru a impune o direcție, o instituție de cercetare.

Profesor pe rînd la Universitățile din Iași și București, la Institutul de Mine, membru al Academiei R.S.R., președinte al Societății de Științe Matematice din R.S.R., membru al Academiei din Bologna, al Academiei din Palermo, al Institutului de filozofie din Paris, a desfășurat o neobosită activitate didactică și academică, strălucind pretutindeni, prin originalitatea, prin larga viziune a științei, dorința de a cuceri și antrena pe tineri.

Termin această solemnă evocare cu gândul pe care l-am exprimat de la început, că Grigore Moisil reușește să rămână permanent în actualitatea noastră științifică, să fie o figură vie a științei românești, să însuflească și astăzi avânturile celor care îl iau ca exemplu și îndrumător în lupta lor pentru afirmare.

Acad. N. TEODORESCU

Gr. C. Moisil — 75 de ani de la naștere

Astăzi se împlinesc 75 de ani de la nașterea prof. Grigore C. Moisil, profesor la facultatea noastră, membru activ al Academiei R. S. România de la reorganizarea sa, președinte al Societății de științe matematice. Nu voi scrie acum despre diferitele și numeroasele activități avute de profesorul, savantul, cetățeanul Moisil (profesor la diferite institute de învățământ superior, ambasador al țării noastre la Ankara, președinte al unor comisii ale Academiei R. S. România, președinte al Uniunii matematicienilor de expresie latină, membru a diferite Academii străine, doctor honoris causa a mai multor universități, membru în comitetele de redacție ale unor publicații prestigioase etc.). Voi scrie despre profesorul Moisil, profesor remarcabil, profesor în adevăratul înțeles al cuvintului, om de știință deosebit, interesat de toate fenomenele de ordin matematic, activ în multiple direcții: algebră, geometrie, analiză, ecuații, aplicații ale matematicii în științele naturii (mecanică, fizică etc.), aplicații ale matematicii în tehnică (mecanisme automate etc.), aplicații ale matematicii în științele umaniste (lingvistică, istorie, arheologie, drept etc.). Prof. Moisil a cultivat cu multă plăcere și pasiune științe de frontieră, având disponibilități neașteptate pentru tot ceea ce era insolit, important nu numai pentru prezent, dar mai ales pentru viitor. A creat adevărate școli în direcții variate (mecanică, mecanisme automate etc.).

Prof. Moisil a fost în permanență tânăr, tânăr ca preocupări științifice, tânăr în ceea ce privește viața matematică din țara noastră, tânăr și totuși cu multă experiență în ceea ce privește viața socială, tânăr înconjurat de tineri cu care s-a înțeles totdeauna foarte bine. Dar și cei care nu mai erau atît de tineri, și mulți dintre noi știm foarte bine acest lucru, rămineau în jurul profesorului, simțind totdeauna o caldă afecțiune, un sprijin puternic, o îndrumare prețioasă din partea sa.

Prof. Moisil își găsea totdeauna timp să stea de vorbă cu cei care doreau să-l vadă. „Telefonează-mi pe la trei și jumătate (la o oră la care n-ai fi îndrăznit să deranjezi pe altcineva) și apoi vii pe la mine”. Iar casa din str. Armenească 14 era permanent deschisă celor ce aveau nevoie de dînsul.

Prof. Moisil a fost o personalitate de mare suprafață atît științifică cît și culturală a țării noastre. Cîți nu așteptau cu nerăbdare „Contemporanul” pentru a citi rubrica „Știință și umanism”, scrisă în spirit alert, într-un limbaj specific, viu, foarte personal, inimitabil. Sau cîți nu sorbeau cuvintele prof. Moisil, atunci cînd apărea la televiziune!... Toți ne amintim vocea sa baritonă, caldă, cu intonații neașteptate, plină de vigoare. Fiind amator de butade și paradoxuri, cuvîntul prof. Moisil te ținea încordat, în așteptarea sfîrșitului neașteptat al unei fraze al cărei început era departe de a-ți spune totul. Plin de viață, iubitor de viață, prof. Moisil era mereu alături de noi toți, înțelegător, dar și intransigent, urmînd o linie de la care nu-l puteai abate niciodată.

Ca unul care a trăit peste douăzeci de ani în preajma profesorului am ținut să menționez cîteva fapte, care — sînt convinși — nu sînt numai prețioase amintiri ale mele dar și ale multora dintre noi.

Conf. P. P. TEODORESCU

*Centrul de calcul al Universității din București, creație a
acad. prof. Gr. C. Moisil*

Printre preocupările de seamă ale Acad. Gr. C. Moisil se înscriu la loc de cinste acelea legate de dezvoltarea învățămîntului, pregătirii cadrelor și cercetării științifice de informatică din țara noastră. În cele ce urmează voi sublinia principalele acțiuni întreprinse de Acad. Prof. Gr. C. Moisil în această direcție.

Astfel, primele cursuri și conferințe legate de știința calculatoarelor conduse de Acad. Moisil au început să se țină la Facultatea de matematică și fizică din București încă din anul 1954, avînd ca obiect teoria algebrică a mecanismelor automate. Începînd din anul 1957, cînd ing. V. Toma a construit la IFA calculatorul CIFA -1, Acad. Moisil a organizat o serie de conferințe, lecții și expuneri despre mașinile de calcul atît în cadrul Facultății cit și în cadrul secției de matematică sau comisiei de automatică a Academiei. Toate aceste acțiuni de pionierat au avut un rol important în dezvoltarea unor cercetări de informatică și creșterea de cadre valoroase de cercetători. Ele au pregătît condițiile care au dus la înființarea în cadrul Facultății de matematică și fizică, în anul 1959, a secției de specializare (anii IV și V) de „Mașini de Calcul”, prima de acest fel din țară, secție creată și condusă timp de peste 12 ani de Acad. Gr. C. Moisil. În cadrul acestei secții s-au predat încă de la început, pe lîngă cursurile de teoria algebrică a mecanismelor automate, mașini de calcul și programarea calculatoarelor și cursuri de programare matematică, lingvistică matematică și altele. În cadrul secției de mașini de calcul, studenții au căpătat în anii de început primele cunoștințe despre limbaje de programare (ALGOL și FORTRAN), programare în cod mașină, descrierea proceselor de calcul, etc. Absolvenții secției de mașini de calcul au constituit cadrele de bază care au demarat activitatea de informatică la noi în țară, mulți dintre ei fiind specialiștii care lucrează azi în diversele centre de calcul.

Următorul pas memorabil realizat de Acad. Moisil pe linia pregătirii cadrelor de informaticieni a fost înființarea în luna februarie 1962 a Centrului de Calcul al Universității din București (CCUB), una din primele unități de informatică creată la noi în țară.

Încă de la început, activitatea CCUB a fost strîns legată de cea a Facultății de matematică și mai ales de secția de „Mașini de Calcul”. Deși la început CCUB avea un personal propriu redus (5 cercetători), în cadrul Centrului au activat majoritatea cadrelor didactice de la catedra de algebră a Facultății precum și numeroși cercetători din afară, în special de la Institutul de Matematică al Academiei. Toți aceștia au beneficiat de îndrumarea atentă a Acad. Moisil, care a condus cu pasiune, devotament și autoritate științifică CCUB timp de aproape 8 ani.

Deși la început CCUB nu a avut nici un fel de calculator, prin grija și perseverența Acad. Moisil, Centrul a fost dotat în 1964 cu un calculator CIFA -3, instalat în localul din str. Ștefan Furtună 125, fapt care a contribuit mult la formarea primilor programatori la noi în țară. Ulterior, (în 1965) CCUB a fost dotat cu calculatoare MEDA 40 TA care au contribuit la inițierea unor ingineri în calculul analogic. Un moment remarcabil în viața CCUB l-a constituit aducerea în anul 1966, în cadrul

unei expoziții pe timp de 6 săptămâni, a calculatorului digital ODRA de construcție poloneză, care a contribuit la îmbogățirea experienței de programare a colaboratorilor Centrului.

Dar cel mai important pas în dezvoltarea CCUB sub conducerea Acad. Gr. C. Moisil s-a făcut în anul 1968 când a fost adus și instalat sistemul de calcul de generația a III-a IBM 360/30, unul din cele mai performante existente la acea dată în țară. În perioada mai 1968 — februarie 1969, acest sistem a fost utilizat în cadrul expoziției de prezentare organizată de firma IBM, cu care ocazie, un număr însemnat de matematicieni, ingineri și economiști din București și din țară, și-au îmbogățit cunoștințele de programare și utilizare a calculatoarelor, beneficiind de asistența specialiștilor străini care lucrau în cadrul expoziției. Acești specialiști țineau diverse cicluri de lecții sau realizau demonstrații practice, de mare utilitate pentru cei care participau la ele.

După închiderea expoziției, sistemul IBM a rămas în dotarea CCUB, în co-proprietate cu Comisia Guvernamentală pentru Dotarea cu Echipamente de calcul și automatizarea prelucrării datelor (organism precursor actualului ICI), și cu Ministerul Agriculturii. În cursul anului 1969 CCUB, la inițiativa Prof. Moisil a organizat 16 cursuri de programarea calculatoarelor, sisteme de operare, lucrul cu fișiere, utilizarea diverselor pachete de programe, unele cursuri fiind organizate pentru ingineri și economiști chiar în diverse ministere sau instituții de cercetare din țară.

Aceste cursuri numite de Prof. Moisil „cursuri libere” au avut un rol deosebit în formarea specialiștilor de informatică; absolvenții acestor cursuri au devenit pionierii activității de informatică ce urma să se dezvolte la noi în țară după anul 1970.

Astăzi, aproape în fiecare Centru de calcul din capitală precum și în unele Centre de calcul din țară există informaticieni care s-au format prin cursuri sau prin stagii de specializare pe lângă CCUB.

Una din menirile importante ale CCUB încă de la înființarea sa, așa cum sublinia în diverse ocazii Acad. Moisil, era aceea de a sprijini învățămîntul în domeniul calculatoarelor din Facultatea de matematică atât prin asigurarea bazei materiale necesare procesului de învățămînt cit și prin propagarea și difuzarea celor mai noi fapte științifice, teoretice și practice din acest domeniu. Ca urmare a acestui deziderat, activitatea CCUB a fost permanent legată de cea a catedrei de algebră condusă de Acad. Moisil. Multe din cursurile ținute în Facultate despre limbajele de programare, sisteme de operare, etc., diversele activități de laborator introduse în planul de învățămînt de Acad. Moisil, au fost acoperite cu personalul CCUB.

Pe lângă rolul de propagator al celor mai noi rezultate din informatică în perioada de pînă în 1970, CCUB, sub conducerea Prof. Moisil a inițiat pentru prima dată la noi în țară cercetări privind unele din aplicațiile noi și neobișnuite pînă atunci ale calculatoarelor și modelării matematice. Menționăm în acest sens cercetările privind utilizarea calculatoarelor în compoziția muzicală, cercetări de recunoașterea formelor și clasificare cu aplicații în diagnosticul automat, aplicații în analiza literară, arheologie, etc.

În același timp, cercetătorii Centrului au continuat și dus mai departe preocupările tradiționale, începute sub îndrumarea Prof. Moisil

în domeniul logicii matematice, teoriei algebrice a mecanismelor automate, lingvisticii matematice și limbajelor formale, algebrei moderne, etc. Acad. Moisil a urmărit și stimulat în permanență activitatea colaboratorilor săi.

Beneficiind de prestigiul și autoritatea sa științifică, Prof. Moisil a militat continuu pentru dezvoltarea CCUB. Astfel în anul 1969, cînd s-a retras de la conducerea Centrului, personalul propriu al acestuia era de 26 lucrători din care 17 cercetători, unii dintre aceștia fiind azi specialiști de prestigiu ai informaticii românești.

Centrul dispunea de un local corespunzător la acea dată, pe care-l menține și astăzi și era dotat la nivel de vîrf cu tehnică de calcul.

Astăzi, cînd comemorăm 75 ani de la nașterea Acad. Gr. C. Moisil, ctitorul CCUB, considerăm că este datoria noastră de onoare să spunem câteva lucruri și despre modul cum a evoluat în anii din urmă ctitoria sa.

Astfel, ca urmare a unui HCM de la sfîrșitul anului 1970 prin care CCUB a căpătat un nou statut juridic, personalul acestuia a crescut pînă azi la 43 cercetători și 25 operatori și tehnicieni. Dotarea Centrului a crescut prin instalarea în 1976 a sistemului de calcul FELIX C-256 precum și prin achiziționarea pe parcurs a diverse echipamente auxiliare, inclusiv echipamente de teleprelucrare. Unele din aceste echipamente auxiliare precum și o parte din personal, funcționează în localul Facultății de matematică pentru a asigura condiții corespunzătoare procesului de integrare a învățămîntului cu cercetarea și producția.

Din punct de vedere organizatoric în CCUB funcționează un laborator de modele și programare, un atelier de proiectare și un colectiv de exploatare a echipamentelor.

Prin creșterea personalului și a dotării s-au creat condiții pentru efectuarea de cercetări în domenii noi ale informaticii. Astfel în CCUB s-au constituit colective care lucrează în domenii ca : simularea discretă, proiectarea limbajelor de programare, analiză numerică, cercetare operațională precum și diverse aplicații ale calculatoarelor în proiectare, în alte științe, etc. Din realizările mai importante ale ultimilor ani menționăm :

- Implementarea pe calculatorul FELIX C-256 a limbajului de simulare SIMUB și a limbajului PLUB pentru scrierea de software ;
- Cercetări care au condus la implementarea unor pachete de programe pentru calculul și gestiunea rezervelor geologice ;
- Dezvoltarea bibliotecii științifice a sistemului FELIX cu programe și subprograme pentru calcule numerice, calcule statistice și optimizări.

Un loc important în activitatea CCUB în ultimii ani îl ocupă sprijinirea procesului didactic al secției de informatică a facultății prin acoperirea unor activități de curs, seminar, laborator și practică productivă.

O realizare de prestigiu a CCUB în ultimii 8 ani o constituie organizarea cursului postuniversitar internațional „Informatică și matematici aplicate” în colaborare cu UNESCO, curs care a fost urmat pînă acum de 91 de cursanți din 31 de țări în curs de dezvoltare.

Printre obiectivele de viitor ale CCUB menționăm instalarea unor echipamente de teleprelucrare în Facultatea de matematică, extinderea

dotării cu minicalculatoare și abordarea unor cercetări privind descrierea matematică a proceselor legate de realizarea sistemelor informatice precum și cercetări privind mărirea performanțelor limbajelor de programare.

CCUB se va strădui să-și aducă contribuția pe măsura posibilităților sale la procesul de punere în slujba omului a informaticii și matematicii, așa cum a făcut în întreaga sa viață creatorul Centrului, Acad. Prof. Gr. C. Moisil.

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Conf. ION VĂDUVA

GR. C. MOISIL, Profesorul

Gr. C. Moisil se trăgea dintr-un lung șir de dascăli și îi plăcea să o spună. El, care era atît de modest, amintea cu vizibilă mîndrie de ascendenții lui, slujitori vrednici ai școlii românești. El însuși a fost Profesor — cuvîntul trebuie scris cu majusculă — așa cum foarte puțini au fost : prin lecțiile ținute la facultate, prin activitatea sa științifică și de asemenea Profesor, la scară socială, a milioane de cititori, ascultători și privitori.

Este foarte greu, poate imposibil, să cuprinzi într-o caracterizare tot ce era de mare valoare în lecțiile profesorului Moisil. De aceea renunț la o asemenea încercare și citez niște cuvinte, rostite acum 15 ani chiar în acest amfiteatru Spiru Haret. Sînt cuvintele lui Gr. C. Moisil despre Dimitrie Pompeiu *, dar care se potrivesc lui Moisil însuși :

„Pompeiu a știut întotdeauna să trezească interesul pentru lucrurile rare. [...] a știut, într-o expunere obișnuită, să introducă lucruri neobișnuite. [...] a știut să ne facă nu numai să înțelegem matematica, ceea ce e de datoria oricărui profesor, dar și să fim obsedați de ceea ce atunci aveam impresia că am putea să înțelegem mai mult. Mare maestru al analogiilor, [...] te învăța un șir foarte mare de lucruri, dar mai ales te învăța să încerci să destrămi orișice idee, orișice demonstrație, ca să pui în evidență și ceea ce era esențial și ceea ce era intuitiv. [...] nu a repetat niciodată lucrurile care se găseau în alte cărți ; găsea că faptul că un student face drumul pînă la facultate îi dă profesorului obligația ca ceea ce studentul ar putea mai bine învăța într-o bibliotecă sau la el acasă, să-l lase să învețe în bibliotecă sau la el acasă. Ceea ce, însă, ascultai de la Pompeiu nu se găsea nici în altă bibliotecă, nici în alt curs, nici la tine acasă, nici în alte lecții — chiar tot ale lui Pompeiu”.

*) Conferința a fost publicată în „Gazeta Matematică“, seria A, vol. 70 (1965), Nr. 3.

Înlocuiți peste tot în acest citat cuvîntul Pompeiu cu Moisil și veți obține o fidelă imagine a profesorului Gr. C. Moisil. Dar îndrăznesc să spun că Moisil a mers mai departe. Crezul său a fost că matematica și aplicațiile ei constituie un tot indestructibil. Ar fi incredibil, dacă n-ar fi adevărat, că el însuși a lucrat și și-a lăsat urma în domenii și probleme atît de deosebite cum ar fi ecuațiile cu derivate parțiale și în special teoria derivatei areolare, mecanica — unde a făcut pionierat aplicînd metode de analiză funcțională —, logicile cu mai multe valori și algebra logicii, teoria algebrică a automatelor, lingvistica matematică și încă altele. Așa cum spunea Mircea Malița, Moisil „a fost mai mult decît un savant, a fost mai mulți savanți întruniți în sesiune permanentă sau luîndu-și locul unul altuia în cicluri succesive mari, reprezentate de teme fundamentale pe care le-a abordat”.

Iată de ce îndrumarea către un orizont vast a fost, poate, dominantă activității profesore a lui Gr. C. Moisil. Era îndemnul lui permanent adresat elevilor și tuturor studenților săi. În mod firesc a extins acest punct de vedere asupra organizării învățămîntului, propunînd ca studenții interesați de anumite probleme de frontieră între matematică și un alt domeniu să poată da examene și la o altă facultate, din domeniul care îi interesează. A propus chiar crearea unor diplome mixte, de matematică și alt profil. (Astăzi, cînd recunoaștem în principiu necesitatea unui învățămînt diversificat și flexibil, numărul cursurilor opționale a scăzut, iar cursurile rămase sînt grupate rigid în așa-numitele pachete de cursuri.)

Cuprinderea vastă a orizontului moisilian depășea măsura obișnuită a matematicianului și uneori intra în panică atunci cînd maestrul îi propunea o ieșire îndrăzneată din domeniul de preocupări cu care începeai să te obișnuiești; de exemplu atunci cînd, algebrist fiind, trebuia să dai un examen de doctorat de mecanică făcută cu metode de topologie algebrică sau un examen de geometrie diferențială. Deși lui Grigore Moisil i se părea firesc să fii universal, este vorba de o stare de rară excepție, după cum o atestă și mărturia unui alt matematician, el însuși o figură proeminentă a culturii românești. L-am numit pe Dan Barbilian, care oferă volumul său, „Teoria aritmetică a idealelor”, cu o dedicație pito-rească, tipic barbiană: „Lui Grigore Moisil, care mă umilește, dar și ferecește, cu întinsu-i eclecticism”, continuată cu cuvinte de caldă recunoștință pentru sprijinul acordat de Moisil în publicarea acestei cărți.

Cert este că dacă polivalența sa este inimitabilă, Gr. C. Moisil a reușit să creeze *mai multe* școli matematice, în diversele domenii în care a lucrat și care sînt prezentate în partea a doua a acestui simpozion. Mai mult — și această situație este unică — a reușit să creeze școală și în domenii în care nu a lucrat personal ca cercetător. În primul rînd este vorba de economia matematică, apoi ne-am putea referi la aplicațiile matematicii în arheologie și altele.

De altfel, ori de cîte ori descoperea pe cineva care avea o idee interesantă, îl încuraja, îl sprijinea, făcea totul pentru punerea în valoare a acelei idei. Generozitatea activă și eficientă cu care a sprijinit pe atîția — elevi sau oameni cunoscuți accidental sau pur și simplu oameni în suferință — este una din trăsăturile sale care ne impresionează cel mai mult și pentru care nu îi vom putea aduce niciodată întregul nostru prinos de recunoștință.

Printre trăsăturile profesorului Pompeiu, evocate de Moisil, mai există una : „Pompeiu putea fi găsit oricînd. Nu oricînd, nu prea de dimineață, dar putea fi găsit în facultate. Marele merit al unui om care crează o școală matematică — și asta am învățat de la Pompeiu — e că era ușor de găsit. Il așteptai uneori, dar știai că vine”. Însă și în această privință Moisil a mers mai departe. Casa lui era mereu deschisă. Te chema să îi arăți ce ai mai lucrat, să îți facă un referat de care aveai nevoie, sau pur și simplu pentru a sta de vorbă. Acolo, în atmosfera caldă a familiei Moisil, te simțeai ca acasă. Orele treceau pe nesimțite și erai fascinat de conversația profesorului. O conversație în care se amestecau discutarea unor chestiuni de matematică, amintiri, glume, sfaturi, idei profunde, unele îmbrăcînd forma celorlalte. Erau și acestea niște lecții de excepție. Plecai de la profesorul Moisil cu regretul că timpul a trecut prea repede, dar în același timp animat de noi elanuri, de dorința de a valorifica în mod cit mai eficient bogata învățătură a maestrului.

Contactul permanent cu mintea excepțională a lui Grigore Moisil îți stîrnea dorința, probabil naivă, de a încerca să deslușești care este secretul gîndirii acesteia atît de eficiente, iar dacă nu poți avea spontaneitatea și scîlpirea butadelor sale, să ai măcar un ghid, un model pentru propria-ți gîndire. Poate că forța principală a gîndirii moisilienne era puterea de disociere între aspecte care în mod curent sînt confundate. Iată, spre exemplificare, cîteva citate din volumul postum „Știință și umanism” care înmănunchiază tabletele publicate în „Contemporanul”. În articolul „Post universitare” se referă la reciclare : „Faptul că omul învață în permanență s-a știut dintotdeauna. Ceea ce e nou în ideea educației permanente este altceva : cheltuielile de școlarizare post-școală intră în prețul de cost : cea mai scumpă știință este mai ieftină decît neștiința”. În tableta „Treaba matematicienilor” se referă la aplicarea matematicii în domeniul umanisticii : „[...] o obiecție sentimentală : cunoașterea matematică a fenomenelor estetice sau psihice le secătuieste, se spune, de tot ceea ce ele au cald, viu, poetic [...]. În aceste afirmări se joacă uneori pe o confuzie care merită să fie subliniată. Atunci cînd analizăm o poezie, indiferent de faptul că această analiză o facem cu mijloace matematice sau într-o proză obișnuită, e altceva decît atunci cînd ne bucurăm de acea poezie. Exclamația cea mai simplă : « O, cît e de frumos ! » e altceva decît trăirea celui frumos”. Acest „e altceva”, rostit apăsător, cu o intonație de neuitat, era una din vorbele preferate ale lui Grigore Moisil. Tot ca ilustrare a puterii de disociere ce caracterizează analizele întreprinse de Moisil, putem aminti ciclul de articole despre politica editorială. Iată încă un citat din tableta despre Copernic : „Sistemul copernican a negat și o altă autoritate ale cărei ravagii sînt mai ascunse : bunul simț”.

Spuneam că o altă dimensiune a vocației profesionale a lui Gr. C. Moisil a fost aceea de a implanta în cercuri foarte largi ideile care-i erau dragi. Se știe foarte bine că rubrica „Știință și umanism” din Contemporanul devenise, după fericita expresie a lui Geo Bogza, locul cu cea mai mare densitate de materie cenușie pe centimetrul pătrat, din paginile revistei, că Grigore Moisil era o apariție foarte căutată la televiziune, că îndrăgea copii și era îndrăgit de ei, că aprecia entuziasmul copiilor și seriozitatea întrebărilor pe care i le puneau aceștia. Că Grigore Moisil a fost îndrăgit

și admirat de o țară întreagă o dovedesc și numeroșii oameni dinăuntru și din afara matematicii care au scris despre Profesorul Gr. C. Moisil cu o emoție care le-a dus rindurile pe treptele unei mari frumuseți. Voi cita numai câteva rinduri din splendida tabletă a Anei Blandiana apărută recent în România literară. Aceste rinduri dovedesc ce bine au fost înțelese lecțiile Profesorului: „De fapt, ceea ce m-a fermecat întotdeauna la el a fost tocmai neobișnuita capacitate de a dezbrăca de podoabe, de a reduce problemele, elementele și fenomenele la valoarea lor reală, ca și cum impecabila, miraculoasă — într-un atit de absurd univers — logică ar fi devenit, în măiastra sa întrebuințare, un fermecat reactiv în stare să restituie — demascator sau reabilitant — lumii adevăratele dimensiuni. Și, ceea ce era cu adevărat neobișnuit, această operație de dezvelire, departe de a fi fost descurajantă și descendentă, era profund optimistă, dătătoare de speranță. Lumina minții sale era atit de puternică și claritatea sa atit de netă, încit revărsarea ei devenea o bucurie în sine și dezvăluirea proporțiilor sale un act purificator, de salvare”.

Azi, după trecerea anilor, se vedește durabilitatea lecțiilor Profesorului. Cel care a avut norocul de a-i fi elev se întreabă adesea, în diverse situații: ce ar fi spus Moisil? și de multe ori are senzația că știe ce trebuie să facă, pentru că ceea ce te-a învățat Profesorul Moisil nu se uită niciodată și se aplică mereu.

Vreau să mai spun doar că există o carte minunată care se cheamă „Un om ca oricare altul” și ni-l prezintă pe copilul Moisil, pe strălucitul tânăr matematician Moisil, pe savantul, profesorul și omul Moisil. Îmi permit numai să polemizez cu titlul acestei cărți: a fost un om ca nimeni altul.

Conf. SERGIU RUDEANU



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